Cogn Affect Behav Neurosci
DOI 10.3758/s13415-014-0317-4

Visuospatial referents facilitate the learning and transfer of mathematical operations: Extending the role of the angular gyrus

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Abstract Different external representations for learning and solving mathematical operations may affect learning and transfer. To explore the effects of learning representations, learners were each introduced to two new operations ($b{\uparrow}n$ and $b{\downarrow}n$) via either formulas or graphical representations. Both groups became adept at solving *regular* (trained) problems. During transfer, no external formulas or graphs were present; however, graph learners' knowledge could allow them to mentally associate problem expressions with visuospatial referents. The angular gyrus (AG) has recently been hypothesized to map problems to mental referents (e.g., symbolic answers; Grabner, Ansari, Koschutnig, Reishofer, & Ebner *Human Brain Mapping, 34*, 1013–1024, 2013), and we sought to test this hypothesis for visuospatial referents. To determine whether the AG and other math (horizontal intraparietal sulcus) and visuospatial (fusiform and posterior superior parietal lobule [PSPL]) regions were implicated in processing visuospatial mental referents, we included two types of transfer problems, *computational* and *relational*, which differed in referential load (one graph vs. two). During solving, the activations in AG, PSPL, and fusiform reflected the referential load manipulation among graph but not formula learners. Furthermore, the AG was more active among graph learners overall, which is consistent with its hypothesized referential role. Behavioral performance was comparable across the groups on *computational transfer* problems, which could be solved in a way that incorporated learners' respective procedures for regular problems. However, graph learners were more successful on *relational transfer problems*, which assessed their understanding of the relations between pairs of similar problems within and across operations. On such problems, their behavioral performance correlated with activation in the AG, fusiform, and a relational processing region (BA 10).

Keywords Learning · Transfer · Math · Angular gyrus · Fusiform · Visuospatial representation

The manner in which information is represented and solved during learning doubtless affects transfer (Kotovsky & Fallside, 1989). In this research, we were interested in the potential benefits of visuospatial referents for mathematical learning and transfer. Learners were each introduced to two new math operations (designated by ↓ and ↑) via either formulas or graphical representations. After acquiring mathematical competence in one of these two ways, the two groups mentally solved familiar and novel problems that were presented in an identical symbolic form across groups. To expand on prior research that had largely been behavioral (for reviews, see Arcavi, 2003; Presmeg, 2006), we collected neural imaging data in order to gather insight about the different patterns of brain activity that arise from differences in instruction representation (formulas vs. graph referents) and how they might support differential transfer.

Benefits of visuospatial information for mathematical learning

Visuospatial processes obviously have a role in spatial domains of mathematics like geometry, but they also appear to have a role in more symbolic domains of mathematics (Arcavi, 2003). Some (but not all) famous mathematicians have reported relying heavily on mental imagery to guide their mathematical thinking (Tall, 2006; see Hadamard, 1945, for a discussion of Einstein). One key feature of effective

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Published online: 10 September 2014

visuospatial math representations (e.g., number lines, strips, and graphs) is that they tend to spatially represent relative magnitudes (e.g., as locations, lengths, or areas). Several studies of mathematical learning have pointed to the relevance of spatially represented numerical magnitudes. For example, children's ability to position numbers on a number line is correlated with their arithmetic ability and overall math achievement (Booth & Siegler, 2006; Geary, Hoard, Byrd-Craven, Nugent, & Numtee, 2007; Siegler & Booth, 2004; Siegler & Pyke, 2013). Interacting with games like Chutes and Ladders that associate numbers with linearly spaced locations enhances children's ability at number line estimation and magnitude comparison (Ramani & Siegler, 2008; Whyte & Bull, 2008) and at learning new arithmetic facts (Siegler & Ramani, 2009). Cartesian graphs are another common example of using spatial layout to represent relationships among numbers, used both for the instruction of older children and to facilitate mature mathematical reasoning.

Visuospatial magnitude representations can also play an active role as learners solve specific problems. For example, Booth and Siegler (2008) found that children were better able to memorize or estimate answers for specific addition facts (e.g., 5 + 4 = 9, 18 + 16 = 34) when, during training, they had been exposed not only to the symbolic fact, but also to shaded bars representing the magnitudes of each operand and the sum. Visuospatial representations have also proved beneficial for solving arithmetic word problems (for a meta-analysis, see Hembree, 1992). For example, students' spontaneous construction and use of effective visuospatial representations can predict their math problem-solving performance (Blatto-Vallee, Kelly, Gaustad, Porter, & Fonzi, 2007; Hembree, 1992; van Garderen, 2006). Such strategies presumably contribute to the correlations found between spatial ability and mathematics performance (e.g., Clements & Battista, 1992; Gathercole & Pickering, 2000; Kyttälä & Lehto, 2008; Reuhkala 2001; for a review, see Mix & Cheng, 2011).

Diagrams are not always beneficial, however (e.g., Berends & van Lieshout, 2009; Booth & Koedinger, 2012; Larkin & Simon, 1987). For example, students (especially those with low math or spatial ability) may generate pictorial representations that depict irrelevant details of word problem objects (e.g., cars, trees) rather than relevant quantitative relations (i.e., schematic or pattern imagery), and the generation or use of irrelevant pictorial images can be unhelpful or detrimental to problem-solving success (Hegarty & Kozhevnikov, 1999; Presmeg, 1997, 2006).

Since some students might be inclined to construct unhelpful images, instruction in constructing effective visuospatial representations can facilitate problem-solving ability (Hembree, 1992; Lewis, 1989). For example, students trained to generate a number-line-like representations to order the quantities and variables in word problems exhibited gain and transfer superior to that among controls (Lewis, 1989). Similar spatial representations of relative quantities with strips or bars are commonly taught and used in countries such as Singapore (Beckmann, 2004) and Japan (Murata, 2008), where students exhibit high math achievement.

In the present research, we contrasted *graph learners*, who were instructed on the relevant visuospatial representations for mathematical expressions, with *formula learners*, who learned how to solve these problems without a spatial referent. In contrast to past research, our focus was not on the use or construction of external diagrams, but rather on learners' subsequent ability to mentally represent and solve problems in a manner informed by the knowledge that they systematically corresponded to visuospatial referents. Thus, the learning representations (formulas or graphs) were never displayed during transfer.

Distinct neural signatures?

Knowledge that problems corresponded to visuospatial referents was expected to invoke distinct mental processes, and thus brain activation patterns, in graph versus formula learners. However, in comparison to behavioral research on visuospatial representations in math learning, there have been relatively few brain-imaging studies. In one of the few studies, K. Lee et al. (2007; see also Terao et al., 2004) explored BOLD activation differences in a context in which participants read relations within a word problem (e.g., James has 50 fewer watches than Mike) and then mentally had to generate either a symbolic representation (J = M – 50) or a visuospatial representation (a short horizontal bar for James, a longer one beneath it for Mike, with the difference labeled "50"). Participants then saw a representation (of the appropriate type) on screen and verified whether or not it corresponded to their mental representation. No group differences emerged in this behavioral representation verification task, and no brain regions were significantly more active in the visuospatial than in the symbolic condition. However, K. Lee et al.'s participants never had to solve the problems per se. In the present study, mental representations (symbolic or graphical) could be constructed as part of the solution process, but we also wanted to capture the processes that utilize such representations to obtain an answer.

Math tasks generally induce activation in prefrontal and parietal areas (e.g., Menon, Rivera, White, Glover, & Reiss, 2000). The majority of research on the neural basis of mathematics has gone into understanding the role of various regions in basic numerical and arithmetic tasks (e.g., Castelli, Glaser, & Butterworth, 2006; Naccache & Dehaene, 2001; Eger, Sterzer, Russ, Giraud, & Kleinschmidt, 2003; Isaacs, Edmonds, Lucas, & Gadian, 2001; Molko et al., 2003; Piazza, Izard, Pinel, Le Bihan, & Dehaene, 2004; Pinel, Piazza, Le Bihan, & Dehaene, 2004). On the basis of evidence from a number of such studies, Dehaene, Piazza, Pinel, and Cohen (2003) emphasized the importance of three parietal regions, the horizontal intraparietal sulcus (HIPS); the posterior superior parietal lobule (PSPL), and the angular gyrus (AG). Dehaene et al. associated these three parietal regions, respectively, with magnitude processing, visuospatial processing/attention, and the retrieval of verbally stored arithmetic facts (e.g., 2 + 2 = 4). Additionally, Dehaene (1997; see also Schmithorst & Brown, 2004) implicated the fusiform in the visual processing of numbers. As discussed below, these four regions, though sometimes with extended functional interpretations, are areas of interest in the present study.

Such research on basic numeracy and arithmetic has often been concerned with routine aspects of math. The present work also builds upon research exploring the neural and behavioral patterns in more complex math contexts, like algebra (Anderson, 2005; Danker & Anderson, 2007; Ravizza, Anderson, & Carter, 2008), and contexts requiring learners to extend their math knowledge beyond the instructed problem types and procedures (Anderson, Betts, Ferris, & Fincham, 2011; H. S. Lee, Fincham, Betts, & Anderson, 2014; Wintermute et al., 2012). In addition to the four regions mentioned above, such work motivated our interest in two prefrontal areas, the lateral inferior prefrontal cortex (LIPFC) and rostrolateral prefrontal cortex (RLPFC), which will be elaborated below.

Besides the manipulation of learning representation, to help us better assess the possible involvement of brain regions in processing visuospatial mental referents, we included two types of transfer problems, computational and relational, which typically differed in visuospatial referent load (one graph vs. two). For graph learners, regions involved in processing visuospatial referents should be sensitive to this referent load manipulation. Thus, with prior mathematical research and the experimental manipulations in mind, we investigated six predefined regions of interest, described below (Table 1).

Angular gyrus (AG) In Dehaene et al.'s (2003) theory of mathematical processing, the left AG is involved in the memory retrieval of well-learned arithmetic facts (e.g., 2 × 2 = 4). This view is compatible with evidence that the AG is more active when solving overlearned versus untrained arithmetic problems (e.g., Delazer et al., 2003; Grabner et al., 2009). Grabner, Ansari, Koschutnig, Reishofer, and Ebner (2013; also Ansari, 2008) have since suggested a broader, though compatible, interpretation of AG function—that the AG is generally involved in semantic mappings from symbols to referents. This view is compatible with evidence that the AG also supports access to semantic meanings for linguistic expressions (e.g., Binder et al., 1997). Thus, for overlearned arithmetic facts, the AG would support mapping a problem to its answer (qua referent)—which is functionally similar to Dehaene et al.'s verbal fact-retrieval role. However, Grabner et al.'s (2013) view can also be extended to math stimuli with visuospatial referents. For our graph learners, although no problems are overlearned to allow direct answer recall, each problem has an associated visuospatial referent.

Thus, one objective of this research was to test the hypothesis that the AG has a role to play in associating mathematical expressions with visuospatial referents. This would be consistent with the idea that the AG may generally support associating symbolic problem expressions with mental referents (Grabner et al., 2013). This hypothesized symbol–referent mapping role for the AG would predict greater AG activation among graph than formula learners. Further, among graph learners AG activity was expected to increase with a problem's referent load. Although formula learners can associate a problem with a formula, the formulas are more procedural than referential in nature.

Horizontal intraparietal sulcus (HIPS) Another math-relevant parietal region is the horizontal intraparietal sulcus (HIPS), which is commonly implicated in numerical magnitude comparison tasks (see Cohen Kadosh, Lammertyn, & Izard, 2008, for a neuroimaging meta-analysis). Behaviorally, magnitude comparison latencies decrease as the distance between the two values increases (Moyer & Landauer, 1967), so people are faster comparing 2 versus 9 than 2 versus 5. This numerical distance effect could indicate that numbers are represented spatially along mental number line, so it is easier

Table 1 Descriptions of predefined brain regions of interest

Region	Talairach (*x*, *y*, *z*) Center	Brodmann Region(s)	Height (mm)	Width = Length (mm)
RLPFC	±34, 47, 8	10	12.8	15.6
LIPFC	±43, 23, 24	9, 46	12.8	15.6
AG	±41, –65, 37	39	12.8	12.5
HIPS[a]	±34, –49, 45	40	12.8	12.5
PSPL	±19, –68, 55	7	12.8	12.5
Fusiform	±42, –60, –8	37	9.6	12.5

RLPFC, rostrolateral prefrontal cortex; LIPFC, lateral inferior prefrontal cortex; AG, angular gyrus; HIPS, horizontal intraparietal sulcus; PSPL, posterior superior parietal lobule. [a] HIPS coordinates are based on the meta-analysis of Cohen Kadosh et al. (2008)

to determine which number is smaller if their locations are farther apart. In view of these and other data, Dehaene et al., (2003, p. 489) proposed that "a nonverbal representation of numerical quantity, perhaps analogous to a spatial map or 'number line', is present in the HIPS of both hemispheres." Extending the view of Deheane et al., the HIPS might participate in processing visuospatial math referents beyond the mental number line. If so, HIPS activation would be higher among graph than formula learners, and would be sensitive to referent load among the former.

Posterior superior parietal lobule (PSPL) Another parietal region, the PSPL, is however, more generally associated with visuospatial processing in a variety of tasks including those that involve attention orienting, spatial working memory, eye movements, reaching, grasping and pointing (e.g., Corbetta, Kincade, Ollinger, McAvoy, & Shulman, 2000; Culham & Kanwisher, 2001; Simon, Cohen, Mangin, Bihan, & Dehaene, 2002). Deheane et al. (2003) suggest that the PSPL may play a role in attentional orientation on the mental number line. More broadly, data from Chen et al. (2006) also implicates the PSPL (but not the HIPS) in supporting mental imagery for other visuospatial math representations during calculation (e.g., an abacus). Thus, perhaps the PSPL might be more likely than the HIPS to have higher activation among graph than formula learners, and to be sensitive to referent load among the former.

Fusiform The fusiform gyrus is a visual processing region that has been implicated in numerical tasks. Dehaene and Cohen (1995; Schmithorst & Brown, 2004) suggest it plays a role in processing visual number forms in math stimuli. The fusiform is also implicated in mental imagery (Ganis, Thompson, & Kosslyn, 2004). For example, D'Esposito et al. (1997) found that left fusiform activity was associated with generating mental images of aurally presented concrete nouns (e.g., horse) relative to a condition in which listeners processed aural abstract nouns (e.g., treaty). The right fusiform is active not only when people observe faces, but when they imagine faces (O'Craven & Kanwisher, 2000). Fusiform regions are also active when people view or imagine other items (e.g., houses and chairs; Ishai, Ungerleider, & Haxby, 2000), and when they mentally transform images (e.g., to see if they are mirror images; Wartenburger, Heekeren, Preusse, Kramer, & van der Meer, 2009).

The fusiform has also been implicated in imagery to support mathematical computation. In a PET study (Zago et al., 2001), learners solved horizontally displayed single-digit (6×4) and multidigit (37×14) multiplication problems. They reported mentally reorganizing the multidigit problems into a vertical representation, and exhibited increased fusiform blood flow relative to reading digit pairs or retrieving single-digit facts. In another study, right fusiform activity was correlated with solvers' self-reported use of visualization strategies to solve arithmetic word problems (Zarnhofer et al., 2013). Thus, we expect higher fusiform activation among graph learners than formula learners and this should increase with referential load.

In addition to the above regions, math-related tasks also tend to involve frontal activation. As was reviewed by Schmithorst and Brown (2004), prefrontal areas are proposed to coordinate the sequencing of processing, holding intermediate results in working memory, and detecting errors (Dehaene, 1997; Dehaene & Naccache, 2001; Dehaene et al., 1996; Shallice & Evans, 1978). We focused on two prefrontal regions, described below.

Lateral inferior prefrontal cortex (LIPFC) The prefrontal cortex generally supports working memory and executive control processes, which are necessary for math problem solving. The LIPFC has been associated with declarative memory retrieval in computational models (e.g., in ACT-R; Anderson et al., 2008), and in other theories of memory (e.g., Thompson-Schill, D'Esposito, Aguirre, & Farah, 1997; Wagner, Paré-Blagoev, Clark, & Poldrack, 2001). In several experiments studying tasks like algebra equation solving and geometry proofs (see Anderson, 2007, for a review), LIPFC activity proved to be the best correlate of student proficiency. We thus expected LIPFC activity to predict performance, but had no specific expectations about group effects.

Rostrolateral prefrontal cortex (RLPFC) This anterior prefrontal region in Brodmann Area 10 may have a metacognitive role, since it has been found to be active when reflecting on task performance (Fleming, Huijgen, & Dolan, 2012) and when learners extend their mathematical knowledge beyond practiced procedures (Anderson et al., 2011; Anderson & Fincham (2014), H. S. Lee et al., 2014; Wintermute, Betts, Ferris, Fincham, & Anderson, 2012). It has been implicated in reasoning about higher-order relationships and analogies (Bunge, Helskog, & Wendelken, 2009; Christoff et al., 2001; Volle, Gilbert, Benoit, & Burgess, 2010; Wendelken, Nakhabenko, Donohue, Carter, & Bunge, 2008), including those involving visuospatial relations (Watson & Chatterjee, 2012). Thus, graph learners were expected to engage the RLPFC for relational transfer problems, which had a high visuospatial referent load of two graph referents, and which probed the relations between them. Such relational processing of visuospatial referents in the RLPFC was expected to benefit (i.e., predict) transfer performance among graph learners.

The present study

To explore the effects of visuospatial learning referents on mathematical learning, we sought to investigate relations between: (1) learning representation and brain activation patterns; (2) learning representation and performance; and (3) activation patterns and performance. We introduced students to two new mathematical operators designated by ↓ and ↑, that mapped pairs of numbers onto values. Students were trained on solving regular problems of the form $b{\uparrow}n = X$ (e.g., $3{\uparrow}2 = X$), where they calculated the value (X) and where b and n were single-digit positive integers. Learners were told how to calculate the values by means of either a formula or a graph representation, as is shown in Fig. 1. The graph representations were staircases. Learners were told that $b{\uparrow}n$ was the area of the n stairs starting with the stair of height b. The up arrow (↑) denotes that one goes up from b, and the down arrow (↓) denotes that one goes down. Graph learners could solve regular problems by calculating the sum of the heights of the stairs. Formula learners efficiently calculated the final values without any graphic or explanation.

Graph learners, who were privy to the visuospatial interpretations for problems, could engage in distinctive processing to mentally associate problems with graphical referents. Thus, we expected increased activation among graph (vs. formula) learners in the fusiform, AG, and HIPS/PSPL, on the basis of prior findings (summarized above) that have implicated these regions, respectively, in mental imagery, semantic association, and mental number line processes.

Different learning representations and activation patterns might not, however, necessarily produce performance differences. Indeed, during a learning phase (Day 1), we expected that both groups would become comparably adept at solving regular problems. However, we also included novel transfer problems in the scanning phase (Day 2) to test the hypothesis that graph learners' ability to associate problem expressions and components with meaningful mental referents might facilitate transfer. Note that the graphical representations spatially represent the magnitudes of both operands and the value[1] (Fig. 1). Prior research has suggested that learning and transfer benefit from knowledge about the magnitudes of problem elements (Siegler & Ramani, 2009) and the magnitude relations among these elements, which characterize the operation (Slavit, 1998).

Thus, after mastering the calculations for regular problems in one representation or the other (Fig. 1), learners were asked to solve transfer problems for which they had to extend their knowledge (exemplified in the Appendix). The two classes of transfer problems, computational and relational, differed in the complexity of the referents (one vs. two graphs) and the degree to which we expected knowledge of the graph referents to facilitate performance.

[1] In the expression $b{\uparrow}n{=}X$, the height of the starting stair reflects b, the horizontal extent n, and the area X (note the similarity to area under the curve, as in calculus).

Computational transfer If both learning conditions comparably equipped learners to solve regular problems, then we hypothesized that learners in both groups might comparably handle *computational transfer problems* that allowed them to leverage the same solution process that they had mastered for regular problems. There were two types of such problems:

1. **Negative operands (e.g., $-4{\uparrow}3$ or $4{\uparrow}{-3}$)** Formula learners could apply the formulas as usual. For graph learners, a negative first operand would still designate the starting column, but it would be left of the origin and would have negative height. They had experienced adding negative height columns during training, when solving problems like $2{\downarrow}6 = X$. In the case of a negative second operand, n, they could infer that it required them to traverse the graph in the other direction than would be appropriate for positive n (e.g., $4{\uparrow}3 = 4 + 5 + 6$, whereas $4{\uparrow}{-3} = 4 + 3 + 2$). Thus, a negative operand problem like $4{\uparrow}{-3}$ would have the same single-graph interpretation as the regular problem $4{\downarrow}3 = X$ (see the bottom graph of Fig. 1b).
2. **Unknown operands (e.g., $X{\downarrow}3 = 9$ or $4{\downarrow}X = 9$)** We assumed that both groups would solve these problems by guess and check—that is, guessing a value for X and then applying their regular procedure. Both groups could use their representations and experience with the training set to come up with reasonable guesses. For instance, simply dividing the value on the right of the equal sign by the given operand would give a ballpark value for the missing operand. Note that unknown-operand problems (e.g., $X{\downarrow}3 = 9$ or $4{\downarrow}X = 9$) share the same single-graph interpretation as the corresponding regular problem ($4{\downarrow}3 = X$; see the bottom graph of Fig. 1b).

Relational transfer A separate set of relational transfer problems were intended to probe learners' understanding of the systematic regularities and relations within and across operations. More concretely, as is exemplified in the Appendix, relational problems typically probe the relation between a pair of similar problems with different operations (e.g., $15{\uparrow}4 = X{\downarrow}4$) or within the same operation (e.g., $20{\uparrow}15 = 21{\uparrow}14 + X$). As is evident from these examples, the referential demands of associating problems to referents were higher for relational (vs. computational) transfer problems, because relational problems include two problem expressions (e.g., $5{\downarrow}3 = 4{\downarrow}2 + X$; see also the Appendix), and thus two graph referents, as is exemplified in Fig. 2.

For basic arithmetic, Grabner et al. (2013) demonstrated that AG activation is increased when an arithmetic stimulus is likely to bring two symbolic referents to mind. We assessed the generality of this effect to visuospatial referent demands. The relations probed by relational transfer problems were expected to be more readily apprehended by graph learners,

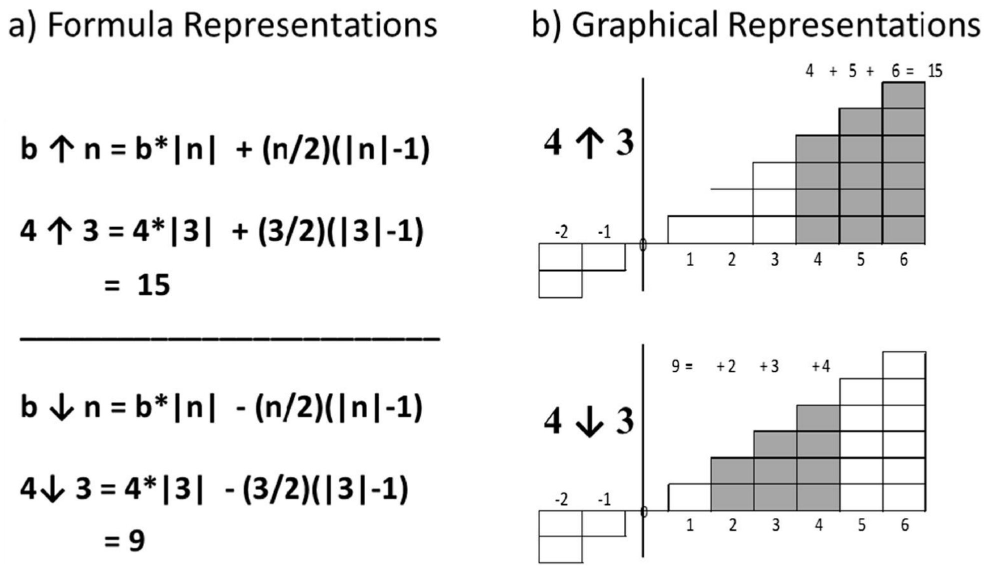

Fig. 1 Example problems to instruct learners how to solve regular problems in (**a**) the formula-learning group and (**b**) the graph-learning group. The on-screen graphs extended from –5 to +15, so as to provide all necessary columns for all problems (b = 2–9, n = 2–6). Graph learners were told that for 4↑3, "4 is the height of the starting column; 3 is the number of columns to sum, rightward, including the starting column." Similarly, for 4↓3, again the starting point was column 4, but the area extended leftward (i.e., the direction in which the stairs went down).

who were not only exposed to all of the problems and answers in the training set, but also to their corresponding graphical representations that rendered the relational information more explicit.

1. **Relating up and down problems (e.g., 5↑3 = *X*↓3)** From the graphical representation in Fig. 2a, we see that problems 5↑3 and 7↓3 both refer to the same area (columns) on the graph; thus, 5↑3 = 7↓3. In the formulas, the two different operations differ in whether their two terms are separated by a + or a – sign (Fig. 1a). We hypothesized that graph representations would provide more intuitive insight into the patterns of relations between up and down problems.
2. **Consecutive-operand problems (e.g., 5↓3 = [4↓2] + *X*)** These problems probe the effects of incremental increases or decreases in the operands. For example, as is shown in Fig. 2b, 5↓3 and 4↓2 both include two columns with heights 4 and 3, and 5↓3 additionally includes a column with height 5, thus 5↓3 = (4↓2) + 5.
3. **Mirror problems (e.g., 5↓5 = *X*↓6, 5↓11 = *X*)** For down-arrow problems, increasing the second operand by one sometimes does not change the final answer. Figure 2c exemplifies that this occurs when the first and second operands are positive and equal (e.g., 5↓5), so incrementing the second operand extends the relevant graph area over the origin, contributing a column of height 0 (area = 0) to the total area. Thus, 5↓5 = 5↓6, or generally, b↓b = b↓(b + 1). As is shown in Fig. 2d, as the second operand becomes larger still, the relevant graph region will include columns left of the origin, which contribute negatively to the sum and cancel the contributions of their counterpart columns to the right of the origin. This cancelation is not quite complete for $n = 2b$ (e.g., 5↓10), because the column 0 lies between the positive and negative columns but does not contribute or cancel anything. Thus, at $n = 2b$ all positive columns will be cancelled except the original one, so that 5↓10 = 5, but 5↓11 = 0. Note that mirror problems (Fig. 2d) implicitly relate two subgraphs to the left and right of the origin, which can be parsed as two areas that sometimes cancel each other out (e.g., 5↓5 + 0↓(5 + 1) = 5↓11 = 0). As is characterized above and in Fig. 2d, this mirror property (Wintermute et al., 2012) may be readily anticipated by graph learners. For formula learners, because the formula for b↓n (Fig. 1) has a structure in which a second term (n/2[|n| – 1]) is subtracted from a first term (b*|n|), it may be foreseeable that for some pair of b and n values, the two terms may cancel to give an answer of 0. However, the visuospatial framework (Fig. 2d) presumably makes this mirror property more salient.
4. **Rule problems (e.g., 5↓*X* = 7↓*X*)** Two special values of the second operand, 0 and 1, yield regularities that accord with simple rules. As in multiplication, when the second operand is 0, the answer is zero (b↓0 = 0 and b↑0 = 0), and when the second operand is 1, the answer is the value of the first operand (b↓1 = b and b↑1 = b). Thus, X = 0 is the solution for 5↓X = 7↓X. From a graphical perspective, n = 0 and n = 1 designate areas that include no columns or just the starting column, respectively. The formulas also allow for the correct computation of answers to rule problems, but the solutions lack the transparency that is available with the graphical interpretation.

Several hypotheses were related to our manipulation of problem type (regular, computational transfer, or relational

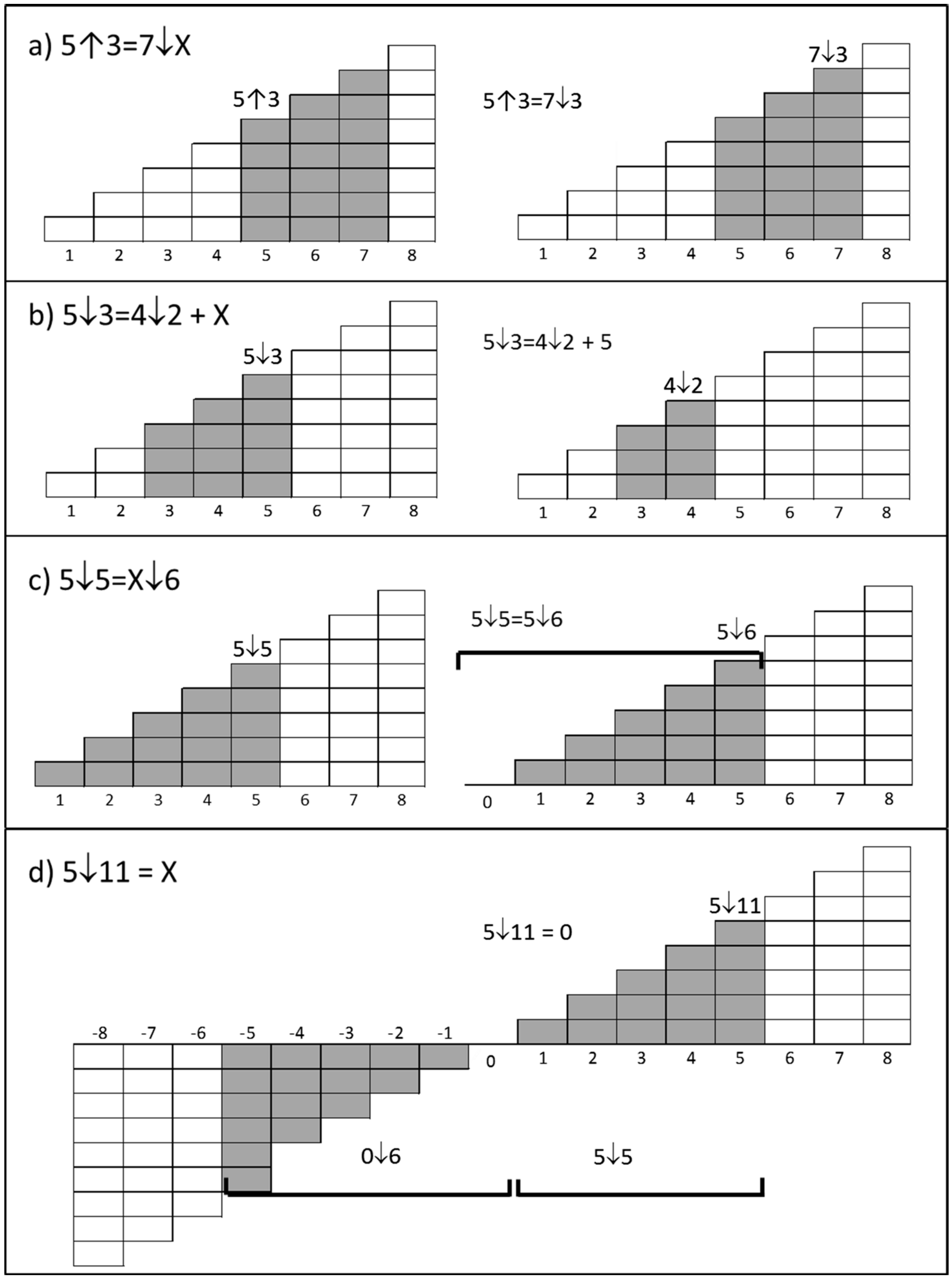

Fig. 2 Illustration of some relationships between graphical referents in relational transfer problems (for more details, see the Appendix): (**a**) relating up and down problems, (**b**) a consecutive-operand problem, (**c**) a mirror-1 problem, and (**d**) a mirror-2 problem

transfer). On behavioral measures, we expected performance to be best on regular problems and poorest on relational transfer problems, which were least similar to the regular (trained) problems. Across learning groups, we expected both groups to achieve comparable proficiency on regular problems. However, we expected better performance on transfer problems among graph learners, because they could mentally characterize problems and their components in terms of meaningful visuospatial referents.

Since relational transfer problems tended to have the highest referent load (two graphs), among graph learners the brain regions allegedly involving semantic and visuospatial referent processing (e.g., AG, HIPS, PSPL, and fusiform) should be most active for these problems.

We also expected the performance benefits of such visuospatial representations, and thus the performance differences across groups, to be highest among relational problems. The Appendix summarizes our assumptions about the solution processes used by the two learning groups for regular and computational transfer problems. For relational transfer problems, the Appendix provides symbolic expressions summarizing the relations in question. However, we expected the solution processes to differ across groups for these problems. For example, to solve $5{\downarrow}3 = 4{\downarrow}2 + X$, on the basis of Fig. 2b, a graph learner need not compute either $5{\downarrow}3$ or $4{\downarrow}2$ to recognize that the only difference between them is due to a single column of height 5 (thus, $X = 5$). Such solving "shortcuts" are not as readily afforded by the visuospatial referents for regular and computational transfer problems. Thus, among graph learners, we expected that relations between visuospatial referent processes (i.e., activations in AG, HIPS, PSPL, and fusiform) and solving performance would be strongest for relational transfer problems.

With no knowledge of graphical referents, a formula learner could also nonetheless solve relational transfer problems like $5{\downarrow}3 = 4{\downarrow}2 + X$ via rote computation (i.e., compute $5{\downarrow}3$ and $4{\downarrow}2$, and then subtract them). Formula learners might also try guessing answers, on the basis of the answer patterns for prior problems gleaned from feedback (e.g., answers were often equal to an operand in the problem). Rote computation and guessing are, however, presumably more error-prone strategies than those that capitalize on the insights afforded by graphical referents (Fig. 2).

Method

Participants

A group of 49 participants (31 male, 18 female; mean age = 23.9, $SD = 5.3$) recruited from the university community were given course credit or payment for their participation. The participants were randomly assigned to one of the two learning conditions (graph vs. formula).

Procedure

The study entailed two 1.5-h sessions on consecutive days. Training was provided in Session 1. Transfer was assessed in Session 2, which took place in a Siemens 3T Verio Scanner.

Training session

Arithmetic pretest To check whether random assignment yielded two groups with roughly equivalent arithmetic skills at experiment outset, participants completed the addition and the subtraction/multiplication subtests from French, Ekstrom, and Price (1963). Each subtest had two sheets with 60 arithmetic questions each and participants had 2 min per sheet to correctly complete as many problems as possible.

Keypad training To prepare learners to operate a numeric response keypad without looking in the scanner, participants practiced using an occluded numeric keypad during training. With practice, when prompted with a character on the screen (i.e., one of the nine digits, +, –, or ENTER), participants ultimately had to press the correct corresponding key within 1.5 s.

Learning phase Each learner was then trained on two novel math operators: designated, respectively, with an up arrow, ↑, and a down arrow, ↓. Learners were assigned randomly to either the learning condition with graphical representations or the condition with formulas (Fig. 1). Training problems were of the form $b{\uparrow}n = X$ and $b{\downarrow}n = X$, where participants solved for X; and b and n were integers from 1 to 9 and 2 to 6, respectively. The 90 unique training problems (45 per operation) were partitioned into eight blocks, the first and last with 12 problems each and the rest with 11 problems each. Each block had about equal numbers of ↑ and ↓ problems.

Problems were presented one at a time on the screen in black font on a white background. Learners pressed ENTER when they had solved the problem, and typed their answers using the occluded numeric keypad. A problem would time out if learners took longer than 30 s to compute an answer and/or longer than 5 s to type it. Otherwise, learners' answers appeared in blue font below the problem. Upon pressing ENTER after their response, learners got feedback that consisted of three elements: (1) Their answer turned green if correct and red if incorrect; (2) the X in the problem was replaced with the correct answer, framed in a box; and (3) as per the learner's condition, a representation for the particular problem appeared under the learner's answer (either a shaded graph or a formula with the appropriate numerical values substituted in). Feedback was displayed for 5 s when the learner's answer was correct and 7 s when incorrect.

In the first and last training blocks (1 and 8), before entering a solution, participants were required to first interact with the representation for their group (graph or formula). In each interactive block, for each operation (↓, ↑), three problems had $b > n$ and three had $b \le n$. On these interactive trials for the graph group, the problem was presented with an unshaded graph. Learners used the mouse to click on the columns relevant to the current problem (e.g., columns with heights 4, 3, and 2 for $4{\downarrow}3 = X$). Clicked columns became shaded in blue. When the column selection was complete, learners pressed ENTER and proceeded to compute and enter their final answer as for noninteractive trials. The feedback elements were the same as for noninteractive trials, with one addition: The correct graph columns were shaded in dark gray as usual, but additionally, the learner's selected columns were outlined in blue, to help learners notice any mismatch between the correct shaded area and their selection.

On interactive blocks for the formula group, an incomplete equation appeared below the problem (see Fig. 3), and the learner's task was to fill in the empty boxes with appropriate values for the current problem, including the

$$4 \downarrow 3 = \square * |\square| \quad \overset{\text{sign}}{\square} \quad \frac{\square}{2}(|\square|-1)$$

Fig. 3 Example of the representation interaction task for the formula group during interactive training blocks: The learners' task was to fill in empty boxes in the formula with appropriate values for the current problem, including the operation sign (+ or –) between the two terms

operation sign (+ or –) between the two terms. The active cursor began in the leftmost empty box and automatically advanced to the next box once the learner typed a value. The learner's values were shown in blue. When ENTER was pressed after the final formula box, the learner then computed and entered the final answer as in a noninteractive trial. The feedback elements were the same as for noninteractive trials, including a correctly completed equation at the bottom of the screen; however, the learner's completed equation also remained visible (with values in blue) above the correct one, to enable learners to check for any mismatch between their values and the correct ones.

Transfer session

To ensure that learners remembered their training, they were asked to either write out the formulas or appropriately shade graphs (as per their condition) for two example problems (1↑6 and 1↓6). Feedback was provided. The remainder of the session took place in a Siemens 3-T Verio Scanner. An initial warm-up block consisted of eight regular problems (*b* from 2 to 9, *n* from 2 to 5), during the structural scan. The remaining eight blocks, which provided the data for the study, each began with a regular warm-up problem (not analyzed), followed by a randomly ordered mix of two regular problems (one up, one down) and eight transfer problems (four computational and four relational; see the Appendix). Problems were presented in the same way as in Blocks 2–7 of training, except that the feedback duration was always 7 s and all participants got the same type of feedback (i.e., correct answers). Computation formulas and graphs were never displayed in the scanner session. Thus, in contrast to training, the physical presentation was identical across groups. To distract participants from thinking about the prior problem and allow brain activity to return to a relatively constant level between problems, a repetition detection task (12 s) was inserted after the feedback on each trial: A fixation cross (3 s) was followed by a series of letters (1.25 s each), and learners were instructed to press ENTER when the same letter appeared twice in a row.

Images were acquired using gradient echo–echo planar image acquisition on a Siemens 3-T Verio Scanner with a 32-channel RF head coil, with a 2-s repetition time (TR), 30-ms echo time, 79° flip angle, and 20-cm field of view. On each TR, 34 axial slices (3.2 mm) were acquired using a 64 × 64 matrix. Voxels were 3.2 mm high by 3.125 × 3.125 mm^2. The anterior commissure–posterior commissure line was on the 11th slice from the bottom.

fMRI analysis

Acquired images were preprocessed and analyzed using AFNI (Cox, 1996; Cox & Hyde, 1997). Functional images were motion-corrected using six-parameter 3-D registration, slice-time centered at 1 s, and normalized such that voxel time series within blocks had mean value of 100. The functional data were then co-registered to a common reference structural magnetic resonance image by means of a 12-parameter 3-D registration and smoothed with a 6-mm full-width-at-half-maximum 3-D Gaussian filter un order to accommodate individual differences in anatomy.

Our primary goal was to understand brain activity (engagement) during correct problem solving and during the feedback period for incorrectly solved problems. Estimates of engagement (beta weights) were obtained by using general linear models (GLM). Separate first-level design matrices were constructed for analyzing the correct and incorrect trials. In each case, the design matrix consisted of seven model variables and a baseline model of an order-4 polynomial to account for general signal drift. Six of the model variables corresponded to the 3 × 2 cells of problem type (three levels: regular, computational transfer, and relational transfer) by period within trial (two levels: solving and feedback). A single additional variable corresponded to the response entry period within a trial, collapsed over problem types. The design matrix regressors were constructed by convolving the boxcar functions of these variables with the standard SPM hemodynamic function (Friston, Ashburner, Kiebel, Nichols, & Penny, 2011). Each GLM yielded seven beta weights per voxel for each participant. Group-level analyses were performed on these first-level beta estimates.

Both whole-brain exploratory analyses and predefined region-of-interest analyses of the average beta weight per region were conducted. As we discussed in the introduction, the predefined regions were the AG, HIPS, PSPL, fusiform, LIPFC, and RLPFC. Their locations are summarized in Table 1. Left and right analogues were used for each region.

Results

Of the 49 participants, nine were excluded from the analyses: Two did not show up for the transfer session, one felt claustrophobic upon entering the scanner and withdrew, one had a brain abnormality identified by the imaging technician, one could not master the occluded keypad during training, and four did not master solving regular problems during training, on the basis of a threshold of 71 % correct in the last four noninteractive blocks (as in Wintermute et al., 2012). Thus, all analyses are based on the 40 participants (20 per group) with complete data (mean age = 23.3, *SD* = 4.9; 26 males, 14 females). Within our analyses of variance (ANOVAs), Bonferroni corrections were used for pairwise comparisons.

Behavioral results

The two learning groups did not differ significantly in age, gender distribution, or performance on the arithmetic pretest (Fs < 1).

Training session Omitting Blocks 1 and 8, in which learners interacted with either a graph or formula (as per their training group) prior to entering an answer, there were six core training blocks, and these were grouped into two halves of three blocks each. Panels a and b of Fig. 4 summarize the accuracy and latency data for the training session. The accuracy and latency (on correct trials) were each analyzed in a Learning Group (formula vs. graph) × Training Half (1st, 2nd) × Difficulty (n value: 2–6) ANOVA. The n value determined the number of additions for the graph learners (i.e., n – 1) and should have a stronger effect on the performance for this group.

Performance improved from the first to the second half of training in both accuracy (78 % to 87 %), $F(1, 38) = 26.26$, $p < .001$, $\eta_p^2 = .41$, and latency (9,218 to 7,321 ms), $F(1, 37) = 79.41$, $p < .001$, $\eta_p^2 = .68$. Formula and graph learners did not exhibit significant overall differences in either accuracy (82 % vs. 84 %; $F < 1$) or latencies (7,978 vs. 8,561 ms; $F < 1$). Furthermore, for accuracy, we found no interaction of group with training half, $F(1, 38) = 1.03$, $p = .318$, $\eta_p^2 = .03$. However, latencies decreased more with training half among formula learners, producing a Group × Half interaction, $F(1, 37) = 15.60$, $p < .001$, $\eta_p^2 = .30$. Formula learners produced answers more slowly than graph learners initially, but by the end of training they were marginally faster. Their reduced latencies may reflect the fact that the number and nature of computation steps are more consistent across problems when computing answers via fixed formulas. Thus, aspects of the computation sequence could become very well practiced and quick to execute.

Overall, as the size of the second operand (n) increased, accuracy decreased, $F(4, 152) = 4.23$, $p = .003$, $\eta_p^2 = .10$, and latencies increased, $F(4, 148) = 109.73$, $p < .001$, $\eta_p^2 = .75$. We also found a strong interaction with group [accuracy, $F(4, 152) = 4.64$, $p = .001$, $\eta_p^2 = .11$; latency, $F(4, 148) = 32.98$, $p < .001$, $\eta_p^2 = .47$]. As expected, the effects of n were stronger for graph learners, for whom n determined the number of additions. Graph learners showed a strong linear relationship between n and latency; regressing latency against n yielded a latency of 3,229 ms at $n = 2$ and a slope of 2,540 ms ($r = .675$, $p < .001$). This pattern provides assurance that graph learners were computing answers in the manner expected for their condition. For formula learners, a regression model analogous to the one for graph learners showed a latency of 5,962 ms for $n = 2$ and a slope of only 863 ms ($r = .290$, $p < .001$), substantially smaller than that for the graph group. When using the formulas (Fig. 1a), the number of computation steps required did not vary with n. However, the time to retrieve or compute answers for individual steps (e.g., "multiply $b*n$") tends to increase with operand size in basic arithmetic (i.e., the *problem size effect*; for a review, see Zbrodoff & Logan, 2005).

In summary, the groups computed answers via different strategies as per their instructed representations, but both groups acquired comparable ability to generate accurate answers.

Transfer session Figures 4c and d summarize the accuracy and latency data, respectively, for the transfer session. For both accuracy and latency (on correct trials), we conducted 2 (learning group: graph vs. formula) × 3 (problem type: regular, computational transfer, relational transfer) ANOVAs. Problem-type effects emerged for both accuracy, $F(2, 76) = 125.95$, $p < .001$, $\eta_p^2 = .77$, and latency, $F(2, 76) = 62.60$, $p < .001$, $\eta_p^2 = .62$: As expected, accuracy was highest and latencies were shortest for regular problems. Accuracy was lower for relational than for computational transfer problems, but the problems' latencies did not significantly differ. Across groups, we found no main effect of latency, $F(1, 38) = 0.27$, $p = .608$, $\eta_p^2 = .01$, but graph learners attained higher overall accuracy in the transfer session, $F(1, 38) = 4.90$, $p = .033$, $\eta_p^2 = .11$. As was reflected in a Group × Problem Type interaction for accuracy, $F(2, 76) = 16.89$, $p = < .001$, $\eta_p^2 = .31$, the higher graph-learner accuracy arose primarily from relational transfer problems, whereas the other problem types did not significantly differ, pairwise, across groups (ps > .152). Group also interacted with problem type for latencies, $F(2, 76) = 5.35$, $p = .007$, $\eta_p^2 = .15$, but no significant latency differences were apparent between the groups for any problem type.

Brain-imaging results

For our predefined regions, we report analyses of the activation patterns during correct solving and during feedback after solution failure (i.e., the solving and feedback analyses were on distinct subsets of trials).[2] We then report exploratory whole-brain analyses to check for other regions exhibiting distinct activations across groups or Group × Problem Type interactions. Finally, we assess relations between activation patterns and behavioral performance.

[2] Also, the solving and feedback intervals were separated by a variable-length (up to 5-s) response phase.

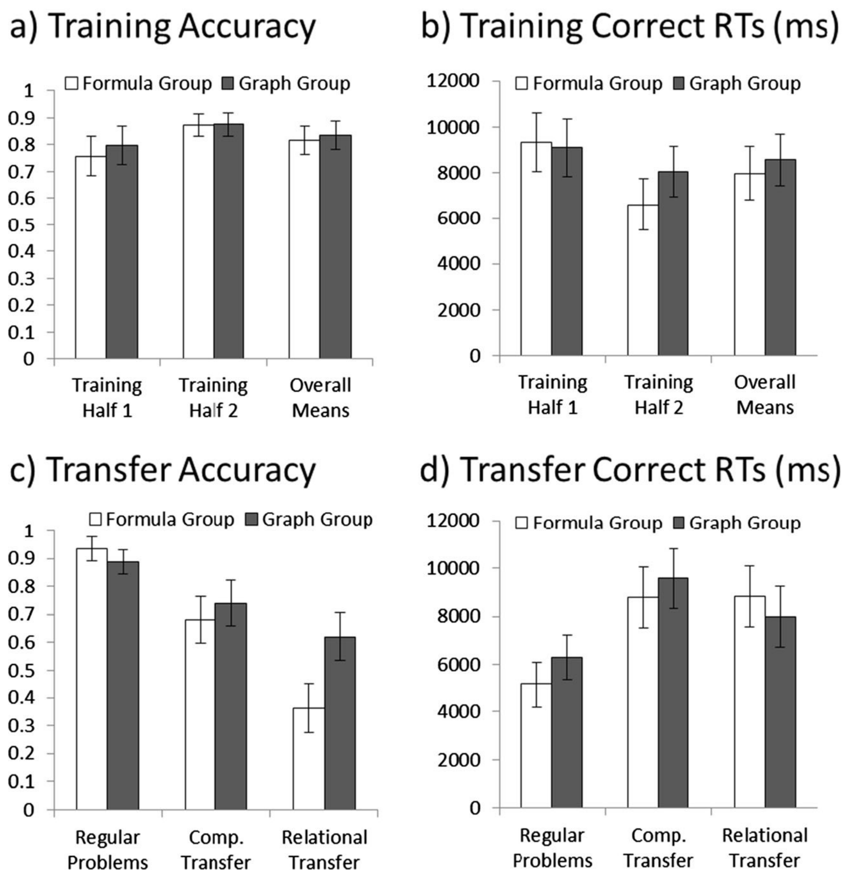

Fig. 4 Behavioral data on (**a**) accuracy during training, (**b**) correct latencies during training, (**c**) accuracy by problem types in the transfer session, and (**d**) correct latencies by problem types in the transfer session. Error bars are 95 % confidence intervals. Latencies on regular problems during transfer are shorter than those in the second half of training, presumably in part because in training the *n* values were 2–6, whereas during transfer they were 2–5

Activations associated with problem solving in regions of interest For each predefined region, we performed a 2 (group: graph vs. formula) × 3 (problem type: regular, computational transfer, relational transfer) × 2 (hemisphere) ANOVA for correctly solved problems. Table 2 summarizes the ANOVA results (corrected for multiple comparisons), and Fig. 5a illustrates the activity patterns across groups and problem types for each region. Our primary interest was in the effects of our learning-group manipulation, including interactions of group with problem type. However, we note that all regions except the PSPL showed main effects of problem type, with regular problems tending to show the least activation, and relational transfer problems the greatest. These analyses are of beta values that reflect the mean levels of engagement during problem solving, and so are corrected for solving time. Thus, as anticipated, our regions of interest were engaged by the demands of the problem-solving task. We also found main effects of hemisphere (left > right) in the RLPFC, LIPFC, and HIPS.

As we hypothesized, the AG exhibited an effect of learning group. During solving, graph learners exhibited higher AG activation than did formula learners. This group difference was significant within each of the three problem types (ps < .028).

Main effects of learning group were not found in the other regions hypothesized to support the processing of visuospatial mental referents—fusiform, HIPS, and PSPL. However, the fusiform, PSPL, and AG[3] exhibited interactions between learning group and problem type. Among graph learners, these regions exhibited significantly higher activation for relational than for computational transfer problems, but among formula learners, this contrast across transfer types was not significant. This increased activation in AG, PSPL, and fusiform for relational (vs. computational) transfer among graph learners had been predicted due to the higher visuospatial referent load for relational transfer problems (two graphs vs. one). Regular problems, like computational transfer problems, were associated with a single graph, and accordingly invoked similar activations in the PSPL and fusiform. However, in the AG, the activations among graph learners were distinct for all three problem types: regular < computational transfer < relational transfer (ps < .001). Though similar in referential load, the computation transfer problems were novel in format relative to regular problems, which may have increased the referential processing demands. (Such novelty may have

[3] For the AG and PSPL, the interaction became marginal after correction for multiple comparisons (uncorrected ps=.015 and .010, respectively).

Table 2 Summary of the analyses of variance on activation patterns during solving (corrects) and feedback processing (incorrects)

	Main Effects						Interactions					
	G: Group		PT: Problem Type		H: Hemisphere		G × PT		G × H		H × PT	
Solving	$F(1, 38)$	η_p^2	$F(2, 76)^a$	η_p^2	$F(1, 38)$	η_p^2	$F(2, 76)^a$	η_p^2	$F(1, 38)$	η_p^2	$F(2, 76)^a$	η_p^2
RLPFC	0.04	.00	**22.75*****	.37	**46.51*****	.55	1.06	.03	4.49	.11	**6.31***	.14
LIPFC	1.92	.05	**34.21*****	.47	**47.01*****	.55	2.73	.07	6.19	.14	2.43	.06
AG	**8.65***	.19	**102.97*****	.73	1.81	.05	**4.84**^{m}	.11	2.29	.06	1.92	.05
HIPS	0.56	.01	**9.76****	.20	**13.48****	.26	4.39	.10	0.59	.02	**6.66***	.15
PSPL	0.69	.02	3.27	.08	1.24	.03	**4.86**^{m}	.11	4.10	.10	**5.42***	.13
Fusiform	0.38	.01	**27.57*****	.42	0.77	.02	**8.81****	.19	2.87	.07	1.46	.04
Feedback	$F(1, 38)$	η_p^2	$F(1, 38)$	η_p^2	$F(1, 38)$	η_p^2	$F(1, 38)$	η_p^2	$F(1, 38)$	η_p^2	$F(1, 38)$	η_p^2
RLPFC	1.48	.04	**5.90**^{m}	.13	**25.74*****	.40	0.27	.01	0.82	.02	1.18	.03
LIFPC	2.35	.06	**9.09***	.19	**11.17***	.23	0.29	.01	1.92	.05	0.80	.02
AG	**6.40**^{m}	.14	**19.04*****	.33	1.66	.04	**7.97***	.17	0.64	.02	0.45	.01
HIPS	3.48	.08	**12.09****	.24	3.92	.09	0.42	.01	0.46	.00	0.44	.01
PSPL	2.11	.05	**19.67*****	.34	0.96	.03	1.91	.05	2.04	.05	0.11	.00
Fusiform	0.99	.03	**40.88*****	.52	1.15	.03	0.08	.00	5.31	.12	0.47	.01

$^{m}p < .01$, $^{*}p < .05$, $^{**}p < .01$, $^{***}p < .001$. The p values were corrected for multiple comparisons for six regions of interest (p = uncorrected $p \times 6$). Significant or marginal F-values are emphasized in bold. [a] If the PT factor violated sphericity (Mauchly's test), Greenhouse–Geisser corrections were also used

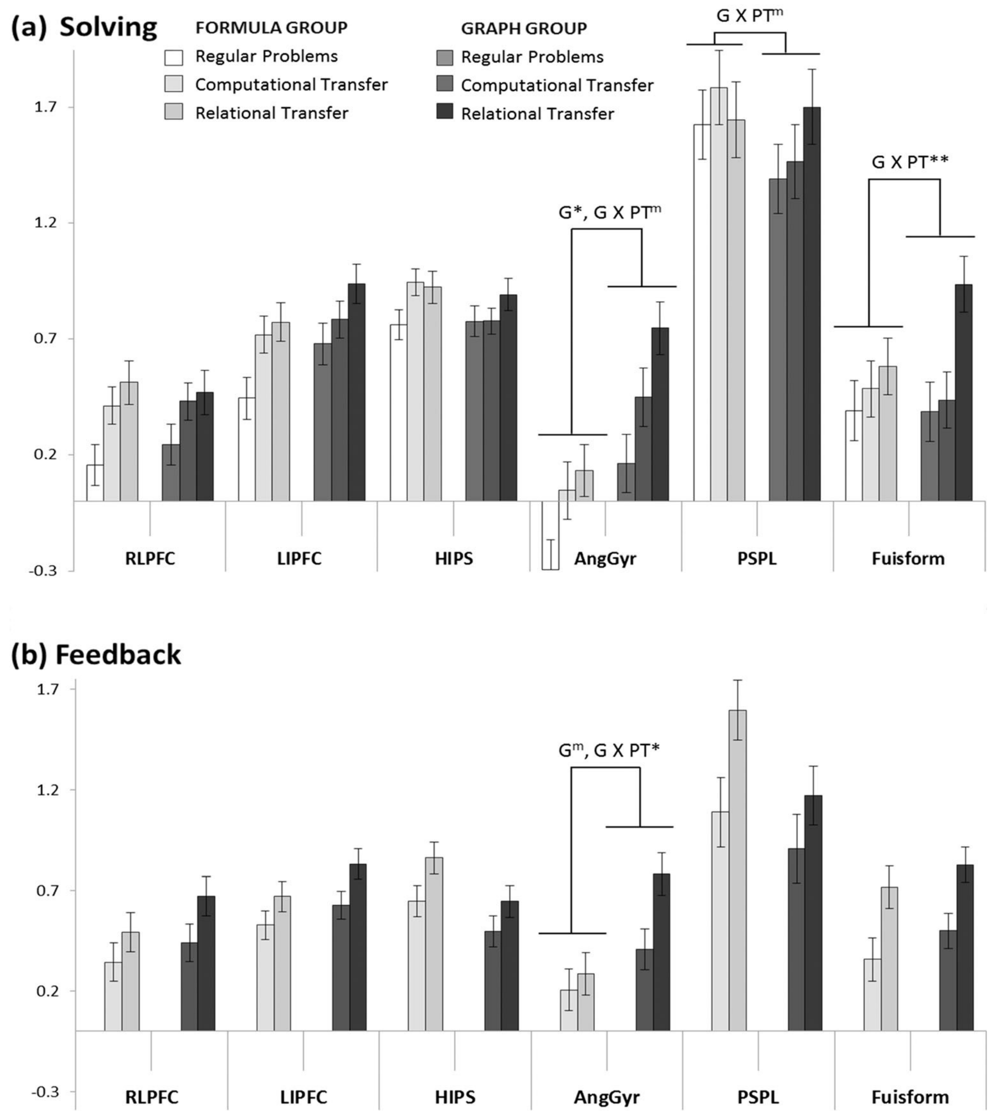

Fig. 5 Percent changes in activation relative to baseline in predefined regions: (**a**) during correct solutions, and (**b**) during feedback after learners had failed to produce a correct solution. G, Group; PT, Problem Type; G × PT, interaction. $^{m}p < .10$, $^{*}p < .05$, $^{**}p < .01$; corrected for multiple comparisons (Table 2). Bars are standard errors

contributed to higher activation on computational transfer than on regular problems, despite their similar demands in both groups in the RLPFC [Fig. 5a], which could play a metacognitive role.) Across groups, although the fusiform did not show a main effect, graph learners did exhibit significantly higher fusiform activation than formula learners when solving relational transfer problems ($p = .046$).

No Group × Hemisphere interactions attained significance. (Thus, the results are collapsed across hemispheres in Fig. 5.) Such interactions would have been of potential interest, because functional roles in a region are sometimes hemisphere-specific. For example, Dehaene et al.'s (2003) model emphasizes the role of the left AG for arithmetic. In the present study, the AG did not show an effect of hemisphere nor any interactions with hemisphere (even prior to multiple-comparison correction), which contrasts with Dehaene et al.'s (2003) emphasis on the left AG.

In summary, the type of learning representation (formula or graph) affected students' subsequent activity patterns when solving regular and transfer problems mentally. Graph learners exhibited higher activity in the AG and, for relational transfer, the fusiform.

Activations associated with feedback processing in regions of interest We hypothesized that differences in mental processes and representations across groups might be evident not only in the activation patterns associated with problem solving, but also in those associated with processing feedback (i.e., the correct answer). When learners produced correct answers, the feedback screen provided no new information. Unsurprisingly, then, preliminary analyses indicated that feedback activations were consistently smaller following correct trials than following trials on which learners had failed to generate a correct response. In the latter case, learners presumably had to reason about why the provided answer was the correct one. In the case of graph-trained learners, such reasoning plausibly involved visuospatial referent imagery. Thus, to explore feedback-induced activation differences, for each predefined region we also performed a 2 (group: graph vs. formula) × 2 (problem type: computational transfer vs. relational transfer) × 2 (hemisphere) ANOVA on feedback processing for incorrect trials. We omitted regular problems from this analysis because relatively few were answered incorrectly. Figure 5b illustrates the feedback activation patterns, and Table 2 summarizes the ANOVA contrasts.

In terms of problem type, in all regions of interest, feedback processing was characterized by more activation following relational than following computational transfer problems. In terms of group differences, feedback patterns in the AG (Fig. 5b) tended to replicate those during problem solving (Fig. 5a). AG activation was higher among graph than among formula learners (marginal after multiple-comparison correction, uncorrected $p = .016$). We also found an interaction of learning group with problem type, such that AG activation was sensitive to the visuospatial referent load among graph learners (computational transfer < relational transfer), but not among formula learners.

Brain-wide analyses Exploratory analyses were also performed to detect other possible regions characterized by either (1) an effect of learning group (graph vs. formula learners) or (2) an interaction of learning group and problem type (regular, computational transfer, and relational transfer).

Using a voxel-wise significance threshold of .001 yielded a brain-wide alpha estimated to be less than .05 by simulation for clusters with more than 20 contiguous voxels (Cox, 1996; Cox & Hyde, 1997). As is detailed in Table 3 and illustrated in Fig. 6, four such regions were found for the graph versus formula group contrast (in all cases, graph > formula), and three regions were found for the Learning Group × Problem Type interaction. The activation patterns in these regions, partitioned by group and problem type, are shown in Fig. 7. The regions identified by a main effect of learning group were assessed for Group × Problem Type interactions via Group × Problem Type ANOVAs, but none proved significant (ps > .20). Similarly, the regions identified by a Group × Problem Type interaction were assessed for main effects of learning group, but none proved significant (ps > .09).

Note that the left parietal region identified to have a main effect of learning group overlapped with the supramarginal and angular gyri (Fig. 6), and the activity patterns (Fig. 7) there were similar to those in the predefined AG region (Fig. 5a). Two of the regions identified in the interaction analysis—the right occipital and left fusiform regions—proved sensitive to visuospatial referent load among graph learners: Activity was higher among relational transfer problems (associated with two mental graphs) than among computational transfer and regular problems (each associated with one mental graph, ps < .001), which did not differ significantly (ps > .350). In contrast, among formula learners, activity in these two regions did not vary significantly with problem type. Note that the left fusiform region from the brain-wide analysis overlaps with our predefined left fusiform region (Fig. 6) and is characterized by a similar activity pattern (Figs. 7 and 5a, respectively). Across groups, activations were higher among graph than among formula learners on relational transfer problems in the right occipital region ($p = .044$) and left fusiform ($p = .002$).

Relations between performance and activation in predefined regions To assess which of our predefined brain regions were most predictive of behavioral performance, we focused on relational transfer problems, which exhibited the most variation in performance across individuals and exhibited a difference in accuracy across groups. It was suggestive that the group differences in relational transfer accuracy (Fig. 4c) co-occurred with group differences in activation in AG, fusiform, and LIPFC (Fig. 5a). However, to better evaluate the relation between regional processing and performance, we computed correlations between the activations and relational transfer performance within each learning group.

As is shown in Table 4,[4] among graph learners, individual performance differences were predicted by AG, fusiform, and RLPFC activations, whereas among formula learners, no correlations reached significance. Directly comparing the strengths of the performance–activation relations across groups (Table 4, Fisher's z) confirmed that RLPFC activity was more strongly related to both accuracy and latencies among graph than among formula learners. Additionally, the fusiform latency relation was marginally stronger among graph than among formula learners.

Within-group (vs. collapsed) correlations were computed because collapsed-group correlations could simply by driven by group differences. Furthermore, it would be unclear how to interpret regional processing in a way that transcended group boundaries (i.e., mediation); formula learners were never privy to graphical referents, so if some regions were involved in referential processing among the graph learners, these regions might be otherwise engaged by formula learners.

Discussion

In the present research, we explored how different instructional representations (formulas vs. graphs) can impact the

[4] Activations were from the left hemisphere (which tended to respond more strongly to the task and to group differences) and included both correct and incorrect trials. We omitted time-out trials, due to their atypical latencies, and their possibly noisy activations if solvers allowed their minds to wander (vs. being engaged in solving the full time).

Table 3 Regions from whole-brain analyses that exhibited a significant effect of learning group (top) or a Learning Group × Problem Type interaction (bottom)

Exploratory Region	Talairach Peak *x*, *y*, *z*	Brodmann Area	Voxels
Regions exhibiting Effect of Learning Group (Graph vs. Formula Learners)[a]			
L. Inferior parietal lobule	–54, –55, 47	40	210
L. Mid/superior temporal gyrus	–60, –26, –2	21/22	56
R. Inferior parietal lobule	48, –49, 47	40	22
R. Middle frontal gyrus	23, –8, 53	6	20
Regions exhibiting Problem Type × Learning Group Interactions[a]			
R. Middle occipital gyrus	–30, 86, –14	19	78
L. Culmen	–17, –54, –10		21
L. Fusiform	–48, –66, –11	19	21

[a] Correction for multiple comparisons: A voxel-wise $p = .001$ yielded a brain-wide alpha estimated to be less than .05 by simulation for cluster sizes >20 contiguous voxels

learning and extension of mathematical operations. Although training equipped both learning groups to solve regular problems, the imaging and behavioral transfer data provide clear evidence of differential processing and learning. In the AG, activations were higher for graph than for formula learners during subsequent mental solving, even for regular and computational transfer problems that were not characterized by behavioral performance differences across groups. This result speaks to the general utility of imaging data to provide information not behaviorally apparent about the possible use of different cognitive representations and strategies (as in Sohn et al., 2004).

Behavioral differences were evident, however, on relational transfer problems. Graph learners solved more of these problems, which probed relationships across pairs of similar problems within and across operations. Knowledge of visuospatial referents for such problem pairs (Fig. 2) was expected to allow graph learners to capitalize on relational processing in the RLPFC. In contrast, formula learners could solve such problems via rote computation, guessing, or shallow rules inferred from prior feedback (e.g., $b{\uparrow}n = b{\downarrow}{-n}$). RLPFC activation significantly predicted relational transfer performance among graph but not formula learners (relations were significantly stronger for the graph group). However, RLPFC activation did not differ overall across groups. Formula learners may have engaged RLPFC equally, but for different processes (e.g., executive functions).

Parietal regions (HIPS, PSPL, and AG) were also expected to reflect distinctive processing across groups. Parietal areas are routinely associated with math tasks (Dehaene et al., 2003) and visuospatial functions (for a review, see Sack, 2009).

PSPL and HIPS: processing visuospatial mental referents beyond the mental number line?

Within mathematics, Dehaene et al. (2003) implicated the HIPS and PSPL in mental number line processes. Thus, their roles might generalize to processing other visuospatial math referents. This expectation was especially strong for the PSPL, which is more generally associated with visuospatial processing, and has been implicated in supporting the visualization of an abacus during calculation (Chen et al., 2006). The HIPS is associated more specifically with the mental representation of numerical quantity in a possibly visuospatial form (mental number line) that may be distinctive (possibly even innate) and might not generalize to processing more general

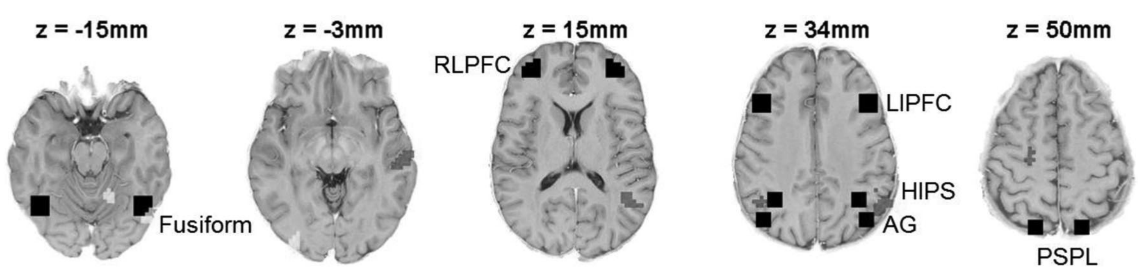

Fig. 6 The six bilateral predefined regions (black) and the regions from brain-wide analyses (Table 3) exhibiting a main effect of learning group (red in electronic figure; graph-learners > formula-learners) or an interaction of Learning Group × Problem Type (yellow in electronic figure). For each brain slice (radiological convention: image left = participant's right), the *z*-coordinate provided is for $x = y = 0$ in Talairach coordinates

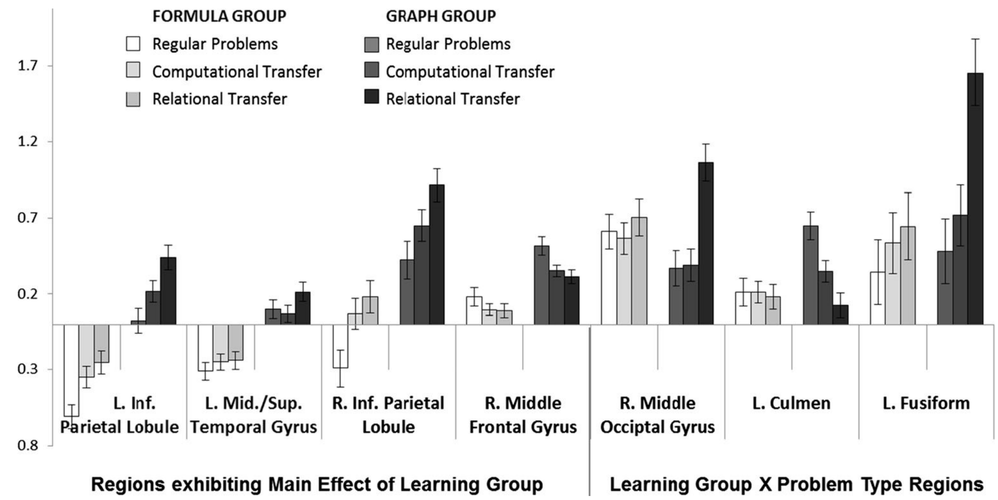

Fig. 7 Percent changes in activation relative to baseline in regions identified in brain-wide analyses as exhibiting a main effect of learning group (left) or an interaction between learning group and problem type (right). Error bars are standard errors

visuospatial math referents. In the present study, these regions did not show main effects of learning representation (formula vs. graph), nor did their activity correlate significantly with relational transfer performance within either group. However, among graph (but not formula) learners, activity in the PSPL was modulated by transfer type (relational vs. computational), the levels of which differed in visuospatial referent load (two graphs vs. one). Thus, our data are compatible with the view that the PSPL may support more general processing related to visuospatial mental referents in math tasks. That said, more clear-cut activation differences across groups were found in the AG and fusiform.

Angular gyrus: associating symbolic math problems with visuospatial referents?

We assessed the hypothesis that the angular gyrus may generally support associating symbolic problem expressions with mental referents (Grabner et al., 2013). Accordingly, the AG was more active among graph learners, who could mentally associate problems with visuospatial referents, than among formula learners, who were not privy to this referential interpretation. The group effect was present in the predefined AG region during solving, and marginally during feedback processing (incorrects). A left inferior parietal region, which overlapped the AG, also emerged in the whole-brain group contrast (Table 3, Fig. 6).

These group differences would not be easily interpretable under a related, but narrower, functional role for the AG: arithmetic fact retrieval (Dehaene et al., 2003). The AG has been especially implicated in the retrieval of multiplication facts such as 3 × 2 = 6 (K. M. Lee, 2000). Since only the formulas explicitly required multiplication (Fig. 1), according to this view one might have expected increased instead of decreased AG activity among formula learners. Increased AG activation among graph learners would not be expected even if AG mediates the retrieval of all basic arithmetic facts, including addition (Grabner et al., 2009). If we consider regular problems, for which the strategies are most clear, note that for the formula group, $2\uparrow4 = 2 \times 4 + 4/2(4 - 1)$, could involve five retrievable subproblems ($2 \times 4 = 8$, $4 - 1 = 3$, $4/2 = 2$, $2 \times 3 = 6$, and $8 + 6 = 14$). For the graph group,

Table 4 Relations between mean activations in the predefined regions and performance for relational transfer

Regions	Accuracy			Correct Latency		
	Graph Learners	Formula Learners	Group Diff? Fisher's z^a.	Graph Learners	Formula Learners	Group Diff? Fisher's z^a
L. RLPFC	**.457***	–.135	**1.83***	**–.546****	.080	**–2.02***
L. LIPFC	.321^m	.054	0.81	–.357^m	–.335^m	–0.07
L. AG	–.132	–.121	–0.74	**–.427***	–.316^m	–0.38
L. HIPS	.228	.298	–0.22	–.360^m	.006	–1.12
L. PSPL	.071	.033	0.11	–.286	–.227	–0.18
L. Fusiform	.100	.006	0.28	**–.634****	–.222	**–1.52**^{m}

^{m} $p < .10$, * $p < .05$, ** $p < .01$ (one-tailed hypothesis: better performance with higher activation). Significant or marginal *F*-values are emphasized in bold. ^{a} Fisher's *z*, to assess whether *r*(graph learners) > *r*(formula learners); see http://vassarstats.net/rdiff.html

2↑4 is the sum of four columns (2 + 3 + 4 + 5) and can be computed by retrieving three facts—that is, first retrieving 2 + 3 = 5, then 5 + 4 = 9, and then 9 + 5 = 14 (or the last sum might be solved using decomposition, 9 + [1 + 4] = 10 + 4; LeFevre, Sadesky, & Bisanz, 1996). So, given that fewer basic fact retrievals are not definitively necessary when applying the formulas, it is not apparent under a fact-retrieval interpretation why AG activity was elevated in the graph group. Furthermore, activations in our predefined AG region were bilateral, rather than restricted to the left hemisphere, as predicted by the verbal fact retrieval view (Dehaene et al., 2003; see also Grabner et al., 2009).

The broader symbol–referent mapping role for the AG is not only compatible with our group effect but also with the finding that among graph learners, AG activations were especially elevated for relational transfer problems, which are associated with two visuospatial referents. On such problems, for which graphical representations might provide insight for solving "shortcuts" (Fig. 2), AG engagement was associated with more-efficient solving.

This symbol–referent mapping view of AG function is a unifying one, because more general evidence suggests that the AG supports cognitive access to semantic meanings for linguistic as well as for mathematical expressions (e.g., Binder et al., 1997). For example, the AG is more active for tasks requiring semantic versus surface processing of a word (Binder, Desai, Graves, & Conant, 2009). The AG adjoins visual, spatial, auditory, and somatosensory association areas, which may make it a good candidate for a high-level integration area (Geschwind, 1965). Notably, several other studies have implicated the AG in visuospatial numerical processing (e.g., Cattaneo, Silvanto, Pascual-Leone, & Battelli, 2009; Göbel, Walsh, & Rushworth, 2001; Zorzi, Priftis, & Umiltà, 2002). For example, TMS of the angular gyrus disrupted performance of a visuospatial search task and a number comparison task (using the putative mental number line; Göbel et al., 2001). On the basis of the AG's location and connectivity (Andersen, Asanuma, Essick, & Siegel, 1990), and of their meta-analysis of 120 semantic studies, Binder et al. (2009) suggested that the AG plays a role as a heteromodal association area for complex information integration and knowledge retrieval. Two other regions that showed group contrasts in the present research, the middle temporal gyrus (from the whole-brain analysis, Table 3, Fig. 6) and the fusiform (Fig. 5), also emerged in Binder et al.'s (2009) meta-analysis and were suggested to play a role in semantic integration and retrieval.

Fusiform: mental imagery in math problem solving?

As we noted in the introduction, the fusiform has been implicated in mental imagery tasks (e.g., D'Esposito et al., 1997; Ishai et al., 2000; Wartenburger et al., 2009), including imagery to reorganize a math problem's format (Zago et al., 2001), and the use of visualization strategies to solve arithmetic word problems (Zarnhofer et al., 2013). Across groups, graph learners exhibited higher fusiform activity than formula learners when solving relational transfer problems. Among graph learners, fusiform activity was sensitive to visuospatial referent load, and predicted solving efficiency. Thus, the fusiform seems to support visualization among graph learners to facilitate their solution of transfer problems.

The fusiform is also assumed to play a role in processing visual number forms in math problem stimuli (Dehaene & Cohen, 1995; Schmithorst & Brown, 2004). However, learners in both our conditions had to process the same visual problem stimuli during transfer, so this perceptual demand could not explain group differences in fusiform activation.

If both the AG and fusiform contribute to learners' use of visuospatial mental referents for symbolic math expressions, then differentiating the exact contribution of each region may remain a question for future research. One possibility is that the AG supports associating the symbolic expression with the information necessary to characterize an appropriate referent, whereas the fusiform may support instantiating the visuospatial representation in the mind's eye. Among formula learners, the fusiform might support visualizing and internally manipulating formulas.

Alternate interpretations

Since a region can support a variety of cognitive functions, one might suggest that increased activity in certain regions (e.g., AG, fusiform) among graph learners may be unrelated to visuospatial referent processing. A transfer advantage may owe to insights from exposure to spatial referents during learning, rather than spatial processing during transfer. However, our interpretation that visuospatial referent processing is at play is based on (1) the experimental manipulation of learning representation, (2) prior evidence that implicates these regions in visuospatial processing, (3) the finding that among graph learners these regions were sensitive to our manipulation of visuospatial referent complexity (computational vs. relational transfer), and (4) that these visuospatial representations provide a strategic advantage for relational transfer, and the mental use of such representations thus predicted the transfer advantage for graph learners.

Although only basic arithmetic was required to produce the answers in both groups, it is the case that the arithmetic computations differed somewhat across groups—for example, on regular problems, latency was more sensitive to changes in the operand n among graph than formula learners. Overall, however, the arithmetic "load" was similar across groups (comparable accuracy and latency on regular and computational transfer problems). Thus, basic arithmetic requirements are unlikely a key source of group differences. However, to further probe the contributions of arithmetic requirements and visuospatial referents to transfer, we are developing a follow-up study to manipulate learning representations while equalizing the arithmetic required.

Pedagogical implications

Mathematics is cumulative, so the utility of mastering an operation for future learning and transfer often requires conceptual understanding that supersedes the strategies sufficient to rotely compute answers. Truly understanding a particular operation, and understanding relations between pairs of operations (e.g., multiplication and division) may require learners to develop an "operation sense" (Slavit 1998). Such an operation sense extends beyond "first-order" knowledge of individual numbers' magnitudes and entails apprehending the systematic patterns and relations that characterize each operation. For example, learners may come to appreciate how the magnitude of the answer typically compares to the magnitudes of the operands in certain ranges, sometimes called a "relation to operands principle" (Dixon, Deets, & Bangert, 2001). Note that, for positive operands b and n, the sum $(b + n)$ will be larger than both operands. Similarly, for our operation, $b{\uparrow}n = X$, X will be larger than both operands. Learners' operation sense for basic arithmetic operations is related to success at more advanced math like algebra (Slavit, 1998). When practicing arithmetic (e.g., addition and subtraction), some aspects of operation sense may develop even in absence of visuospatial representations (number lines, graphs, shaded shapes). However, learning representations that render magnitude relations more explicit (graphs vs. formulas) presumably facilitate learners' apprehension of meaningful magnitude patterns.

Our data demonstrate the general utility of visuospatial referents to facilitate operation sense and transfer. The imaging data suggest that the recruitment of visuospatial and semantic regions, even in the absence of external representations, may help explain correlations between mathematical and spatial ability. In contrast to abstract symbol manipulation, visuospatially mediated mental solution processes may facilitate learning by allowing learners to capitalize on the more evolutionarily developed visuospatial processes and circuits.

Our operations can be interpreted as instances of integral calculus (area under the curve), but the implications readily generalize to other mathematical operations. Similar 2-D graphics could be adapted to facilitate students' apprehension of relations in multiplication (e.g., each axis represents an operand and the area is the product) and quadratics (Hoong et al., 2010). Linear representations (1-D) might suffice for subtraction and addition (e.g., Booth & Siegler, 2008). In this era of computer tutors and educational software, which can automate and customize representations for each problem, educators and students may be in a good position to capitalize on such relevant visuospatial representations (Goldin & Kaput, 1996; Healy & Hoyles, 1999).

Conclusions

Mathematics learning is influenced by the type of representation used to define an operation. Visuospatial learning representations that spatially represent the relative magnitudes of operands and answers presumably enable learners to later capitalize on visuospatial mental representations and strategies. On the basis of the superior performance of graph learners on relational transfer problems, visuospatial referents seem to promote the apprehension of relations between similar problems within and across operations (Fig. 2), and presumably foster insight into the patterns and principles characterizing the operation or function as a whole ("operation sense"). Our data contribute support for extending the fusiform's role in math problem solving beyond processing number stimuli (Dehaene & Cohen, 1995) and supporting imagery for symbolic expressions (Zago et al., 2001), to also supporting imagery for visuospatial referents. Our data also support extending the role of the AG: Beyond having a role in accessing answers to overlearned arithmetic facts (Dehaene et al., 2003), we extended Grabner et al.'s (2013) symbol–referent mapping view to suggest that the AG is involved in associating problems with visuospatial mental referents. This general role in semantic association for the AG may help unify the interpretations of linguistic (Binder et al., 2009) and mathematical studies (Dehaene et al., 2003; Grabner et al., 2009). In the present study, we manipulated whether or not learners were privy to a visuospatial referent interpretation for the problems. However, these brain activation patterns might be useful in future research to help ascertain whether or not learners are bringing visuospatial referents to bear in problem solving.

Author note Support for this research was provided by the National Science Foundation, Grant No. DRL-1007945 to J.R.A.

Appendix

Table 5 Types of problems presented during the transfer session in the scanner on Day 2

Problem Type	Example Problems	Principle and/or Solution Process
Regular (16 items/subj)	$3{\downarrow}2 = X$ $3{\uparrow}2 = X$	See Fig. 1 for solution examples for formula learners (Fig. 1a) and graph learners (Fig. 1b)
Computational Transfer		
Unknown *b* (eight items)	$X{\downarrow}2 = 5$ $X{\uparrow}2 = 7$	Learners may guess *X* value then check it, by computing answer in same way as for regulars
Unknown *n* (eight items)	$3{\downarrow}X = 5$ $3{\uparrow}X = 7$	Learners may guess *X* value then check it, by computing answer in same way as for regulars
Negative *b* (eight items)	$-3{\uparrow}2 = X$ $-2{\downarrow}X = 6$	Formulas apply as normal; for the graph, the starting column is to the left of the origin
Negative *n* (eight items)	$3{\uparrow}{-2} = X$ $2{\downarrow}{-5} = X$	Formulas apply as normal; for graph, infer $n < 0$ reverses direction to travel along horizontal axis
Relational Transfer		
Relating Up and Down Problems (eight items)	$31{\uparrow}4 = 31{\downarrow}X$	For positive integers b & n, UpDown-1: $b{\downarrow}n = b{\uparrow}(-n)$ and $b{\uparrow}n = b{\downarrow}(-n)$
	$19{\downarrow}4 = X{\uparrow}4$	UpDown-2: $b{\downarrow}n = (b - n + 1){\uparrow}n$
Consecutive Operand Problems (eight items)	$35{\downarrow}3 = (34{\downarrow}2) + X$ $26{\downarrow}15 = (X{\downarrow}14) + 26$	For positive integers, b & n, Consecutive-1: $b{\downarrow}n = (b - 1){\downarrow}(n - 1) + b$ Task: Solve for the final constant value to add Consecutive 2: $b{\downarrow}n = (b - 1){\downarrow}(n - 1) + b$ Task: Solve for an n or b value.
Mirror Problems (eight items)	$5{\downarrow}5 = X{\downarrow}6$ $9{\downarrow}X = 9{\downarrow}(X + 1)$	Mirror-1: $b{\downarrow}b = b{\downarrow}(b + 1)$, for positive integer b The origin column (area = 0) contributes nothing
	$50{\downarrow}100 = X$ $30{\downarrow}61 = X$	Mirror-2: $b{\downarrow}2b = b$ and $b{\downarrow}(2b + 1) = 0$, for positive integer b If the area crosses the origin, negative columns cancel corresponding positive columns.
Rule Problems (eight items)	$5{\downarrow}X = 7{\downarrow}X$	0 Rule: For any integer b, $b{\uparrow}0 = 0$ and $b{\downarrow}0 = 0$
	$5{\uparrow}2 = 11{\uparrow}X$	1 Rule: For any integer b, $b{\uparrow}1 = b$ and $b{\downarrow}1 = b$

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