CHAPTER 2

# An Iterative Bayesian Algorithm for Categorization

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## 1. Introduction

A rational analysis (Anderson, 1990) is an attempt to specify a theory of some cognitive domain by specifying the goal of the domain, the statistical structure of the environment in which that goal is being achieved, and whatever computational constraints the system is operating under. The predictions about the behavior of the system can be derived assuming that the system will maximize the goals it expects to achieve while minimizing expected costs where expectation is defined with respect to the statistical structure of the environment. This approach is different from most approaches in cognitive psychology because it tries to derive a theory from assumptions about the structure of the environment rather than assumptions about the structure of the mind.

We have applied this approach to human categorization and have developed a rather effective algorithm for categorization. The analysis assumes that the goal of categorization is to maximize the accuracy of predictions about features of new objects. For instance, one might want to predict whether an object will be dangerous or not. This approach to categorization sees nothing special about category labels. The fact an object might be called a tiger is just another feature one might want to predict about the object.

## 2. The Structure of the Environment

It is an interesting question what kind of structure we can assume of the environment in order to drive prediction. The theory developed rested on the structure of ~~natural kind~~ categories produced by the phenomenon of species. Species form a nearly disjoint partitioning of the natural objects because of the inability to interbreed. Within a species there is a common genetic pool which means that individual members of the species will display particular feature values with probabilities that reflect the proportion of that phenotype in the population. Another useful feature of species structure is that the display of features within a freely-interbreeding species is largely independent. Thus, there is little relationship between size and eye color in species where those two dimensions vary. Thus, the critical aspects of speciation is the disjoint partitioning of the object set and the independent probabilistic display of features within a species.

living thing

An interesting question is whether other types of objects display these same properties. Another common type of object is the artifact. Artifacts approximate a disjoint partitioning but there are occasional exceptions–for instance, mobile homes which are both homes and vehicles. Other types of objects (stones, geological formations, heavenly bodies, etc.) seem to approximate a disjoint partitioning but here it is hard to know whether this is just a matter of our perceptions or whether there is any objective sense in which they do. One can use the understanding of speciation for natural kinds and understanding of the intended function in manufacture in the case of artifacts to objectively assess the hypothesis of a disjoint partitioning.

We have taken this disjoint, probabilistic model of categories and used it as the understanding of the structure of the environment for doing prediction about object features. To maximize the prediction of features of objects we need to induce a disjoint partitioning of the object set into categories and determine what the probability of features will be for each category. The ideal prediction function would be described by the following formula:

$$Pred_{ij} = \sum_{x} P(x|F_n) Prob_i(j|x) \tag{1}$$

where $Pred_{ij}$ is the probability an object will display a value $j$ on a dimension $i$ which is not observed for that object, the summation is across all possible partitionings of the $n$ objects seen into disjoint sets, $P(x|F_n)$ is the probability of partitioning $x$ given the objects display observed feature structure $F_n$, and $Prob_i(j|x)$ is the probability the object in question would display value $j$ in dimension $i$ if $x$ were the partition. The problem with this approach is that the number of partitions of $n$ objects grows exponentially as the Bell exponential number (Berge, 1971). Assuming that humans cannot consider an exponentially exploding number of hypothesis we were motivated to explore iterative algorithms such as those developed by Fisher (1987) and Lebowitz (1987).

The following is a formal specification of the iterative algorithm:

1. Before seeing any objects, the category partitioning of the objects is initialized to be the empty set of no categories.
2. Given a partitioning for the first $m$ objects, calculate for each category $k$ the probability $P_k$ that the $m+1$st object comes from category $k$. Let $P_0$ be the probability that the object comes from a completely new category.
3. Create a partitioning of the $m+1$ objects with the $m+1$st object assigned to the category with maximum probability.
4. To predict value $j$ on dimension $i$ for the $n+1$st object calculate

$$Pred_{ij} = \sum_k P_k P(ij|k) \tag{2}$$

   where $P_k$ is the probability the $n+1$st object comes from category $k$ and $P(ij|k)$ is the probability of displaying value $j$ on dimension $i$.

The basic algorithm is one in which the category structure is grown by assigning each incoming object to the category it is most likely to come from. Thus, a specific partitioning of the objects is produced. Note, however, that the prediction for the new $n+1$st object is <u>not</u> calculated by determining its most likely category and the probability of $j$ given that category. Rather, the calculation is performed over all categories. This gives a much more accurate approximation to the ideal $Pred_{ij}$ because it handles situations where the new object is ambiguous between multiple categories. It will weight approximately equally these competing categories.

The algorithm is not guaranteed to produce the maximally-probable partitioning of the object set since it only considers partitionings that can be incrementally grown. It also does not weight multiple possible partitionings as the ideal algorithm would. In cases of strong category structure, there will be only one probable partitioning and the iterative algorithm will uncover it. In cases of weak category structure, it will often fail to obtain the ideal partitioning, but still the predictions obtained by Equation 2 closely approximate the ideal quantity because of the weighting of multiple categories. As we will see, the correlations are about 0.95 between the predictions of our algorithm and the ideal quantities in cases of small data sets.

It remains to come up with a formula for calculating $P_k$ and $P(ij|k)$. Since $P(ij|k)$ proves to be involved in the definition of $P_k$, we will focus on $P_k$. In Bayesian terminology $P_k$ is a posterior probability $P(k|F)$ that the object belongs to category $k$ given that it has feature structure $F$. Bayes formula can be used to express this in terms of a prior probability $P(k)$ of coming from category $k$ before the feature structure is inspected and a conditional probability $P(F|k)$ of displaying the feature structure $F$ given that it comes from category $k$:

$$P_k = P(k|F) = \frac{P(k)P(F|k)}{\sum_i P(i)P(F|i)} \tag{3}$$

where the summation in the denominator is over all categories $i$ currently in the partitioning including the potential new one. This then focuses our analysis on the derivation of a prior probability $P(k)$ and a conditional probability $P(F|k)$.

## 2.1 Prior Probability

With respect to prior probabilities the critical assumption is that there is a fixed probability $c$ that any two objects come from the same category and this probability does not depend on the number of objects seen so far. This is called the coupling probability. If one takes this assumption about the coupling probability between two objects being independent of the other objects and generalizes it, one can derive a simple form for $P(k)$ (See Anderson, 1990, for the derivation):

$$P(k) = \frac{cn_k}{(1-c)+cn} \tag{4}$$

where $c$ is the coupling probability, $n_k$ is the number of objects assigned to category $k$ so far, and $n$ is the total number of objects seen so far.

Note for large $n$ this closely approximates $n_k/n$ which means that we have a strong base rate effect in these calculations with a bias to put new objects into large categories. Presumably the rational basis for this is apparent.

We also need a formula for $P(0)$ which is the probability that the new object comes from an entirely new category. This is

$$P(0) = \frac{(1-c)}{(1-c)+cn} \tag{5}$$

For large $n$ this closely approximates $(1-c)/cn$ which is again a reasonable form–i.e., the probability of a brand new category depends on the coupling probability and number of objects seen. The greater the coupling probability and the more objects, the less likely it is that the new object comes from an entirely new category.

The impact of the coupling parameter $c$ will be to influence the number and size of categories formed. The larger the value, the fewer and larger the categories that will be formed. Since computation costs are linearly related to number of categories and not to size of categories, there might be some pressure to set $c$ larger than its true value in the environment.

One consequence noted of Equations 4 and 5 is that there is a bias to put objects into large categories. Some have questioned the rationality of this. However, it needs to be stressed that Equation 4 just sets the priors and has to be combined with conditional probabilities for Equation 3. If an instance much better matches a smaller category, the conditional probabilities for the smaller category will be much higher and the instance will be assigned to the smaller category. Thus, the bias in Equations 4 and 5 does not mean that such evidence will be ignored. However, if such feature-matching evidence is equivocal, the system will assign the instance to the larger category which is the sensible thing to do.

This base rate effect contributes to the order sensitivity of our algorithm. Suppose we have an instance that is ambiguous between two categories. If by chance we have seen more instances of one category before the instance, we will be biased to assign it to that category. This will make that category larger and increase tendency to assign instances to the category. In some cases of ambiguous stimuli this can snowball.

## 2.2 Conditional Probability

We can consider the probability of displaying features on various dimensions given category membership to be independent of the probabilities on other dimensions. Then we can write

$$P(F|k) = \prod_i P(ij|k) \tag{6}$$

where $P(ij|k)$ is the probability of displaying value $j$ on dimension $i$ given that one comes from category $k$.

This independence assumption does not prevent us from recognizing categories with correlated features. Thus, we may know that being black and retrieving sticks are features found together in labradors. This would be represented by high probabilities of the stick-retrieving and the black features in the labrador category. What the independence assumption prevents us from doing is representing categories where values on two dimensions are either both one way or both the opposite. Thus, it would prevent us from recognizing a single category of animals which were either large and fierce or small and gentle, for instance. However, this turns out not to be a very serious limitation. What our algorithm does in this case is to spawn a different category to capture each two-feature combination–it would create a category of large and fierce creatures and another category of small and gentle creatures.

The effect of Equation 6 is to focus us down on an analysis of the individual $P(ij|k)$. Derivation of this quantity is itself an exercise in Bayesian analysis. We will treat separately discrete and continuous dimensions.

## 2.3 Discrete Dimensions

The basic Bayesian strategy for doing inference along a dimension is to assume a prior distribution of values along the dimension, determine the conditional probability of the data under various possible values of the priors, and then calculate a posterior distribution of possible values. The common practice is to start with a rather weak distribution of possible priors and as more and more data accumulates come up with a tighter and tighter posterior distribution.

In the case of a discrete dimension, the typical Bayesian analysis (Berger, 1985) is to assume that the prior distribution is a Dirichlet density. For a dimension with $m$ values a Dirichlet distribution is characterized by $m$ parameters $\alpha_j$. We can define $\alpha_o = \sum_j \alpha_j$. The mean probability of the $j$th value is $p_j = \alpha_j/\alpha_o$. The value $\alpha_o$ reflects the strength of belief in these priors probabilities, $p_j$. The data after $n$ observations will consist of a set of $C_j$ counts of observations of value $j$ on dimension $i$. The posterior distribution of probabilities is also a Dirichlet distribution but with parameters $\alpha_j + C_j$. This implies that the mean expected value of displaying value $j$ in dimension $i$ is $(\alpha_j + C_j)/\sum(\alpha_j + C_j)$. This is $P(ij|k)$ for Equation 6:

$$P(ij|k) = \frac{C_j + \alpha_j}{n_k + \alpha_0} \tag{7}$$

where $n_k$ is the number of objects in category $k$ which have a value on dimension $i$ and $C_j$ is the number of objects in category $k$ with the same value as the object to be classified. For large $n_k$ this approximates $C_j/n_k$ which one frequently sees promoted as the rational probability. However, it has to have this more complicated form to deal with problems of small samples. For instance, if one has just seen one object in a category and it has had the color red, one would not want to guess that all objects are red. If we assume these are seven colors and all the $\alpha_j$ were 1, the above formula would give 1/4 as the posterior probability of red and 1/8 for the other six colors unseen as yet.

### 2.4 Continuous Dimensions

Application of Bayesian inference schemes to continuous dimensions is more problematic but there is one approach that appears most tractable (Lee, 1989). The natural assumption is that the variable is distributed normally and the induction problem is to infer the mean and variance of that distribution. In standard Bayesian inference methodology we must begin with some prior assumptions about what the mean and variance of this distribution is. It is unreasonable to suppose we can know in advance precisely what either the mean and variance will be. Our prior knowledge must take the form of probability densities over possible means and variances. This is basically the same idea as in the discrete case where we had a Dirichlet distribution giving priors about probabilities of various values. The major complication is the need to state separately prior distributions for mean and variance.

The tractable suggestion for the prior distributions is that the inverse of the variance $\Sigma^2$ is distributed according to a chi-square distribution and the mean has a normal distribution. Given these priors, the posterior distribution of values, $x$, on a continuous dimension $i$ for category $k$, after $n$ observations has the following $t$ distribution:

$$f_i(x|k) \sim t_{a_i}\left(\mu_i, \sigma_i\sqrt{1 + 1/\lambda_i}\right) \tag{8}$$

The parameters $a_i$, $\mu_i$, $\sigma_i$, and $\lambda_i$ are defined as follows:

$$\lambda_i = \lambda_0 + n \tag{9}$$

$$a_i = a_0 + n \tag{10}$$

$$\mu_i = \frac{\lambda_0\mu_0 + n\overline{x}}{\lambda_0 + n} \tag{11}$$

$$\sigma_i^2 = \frac{a_0\sigma_0^2 + (n-1)s^2 + \frac{\lambda_0 n}{\lambda_0 + n}(\overline{x} - \mu_0)^2}{a_0 + n} \tag{12}$$

where $\overline{x}$ is the mean of the $n$ observations and $s^2$ is their variance. These equations basically provide us with a formula for merging the prior mean and variance, $\mu_0$ and $\sigma_0^2$, with the empirical mean and variance, $\overline{x}$ and $s^2$, in a manner that is weighted by our confidences in these priors, $\lambda_0$ and $a_0$.

Equation 8 for the continuous case describes a probability density which serves the same role as Equation 7 for the discrete case which describes a probability. The product of conditional probabilities in Equation 6 will then be a product of probabilities and density values. Basically, Equations (6), (7), and (8) give us a basis for judging how similar an object is to the category's central tendency.

### 2.5 Conclusion

This completes our specification of the theory of categorization. Before looking at its application to various empirical phenomena a word of caution is in order. The claim is not that the human mind performs any of the Bayesian mathematics that fills the preceding pages. Rather the claim of the rational analysis is that, whatever the mind does, its output must be optimal. The mathematical analyses of the preceding pages serve the function of allowing us, as theorists, to determine what is optimal.

A second comment is in order concerning the output of the rational analysis. It delivers a probability that an object will display a particular feature. There remains the issue of how this relates to behavior. Our basic assumption will only be that there is a monotonic relationship between these probabilities and behavioral measures such as response probability, response latency, and confidence of response. The exact mapping will depend on such things as the subject's utilities for various possible outcomes, the degree to which individual subjects share the same priors and experiences, and the computational costs of achieving various possible mappings from rational probability to behavior. These are all issues for future exploration. What is remarkable is how well we can fit the data simply assuming a monotonic relationship.

## 3. Application of the Algorithm

We have applied the algorithm to a number of examples to illustrate its properties. The predictions of this algorithm are potentially order sensitive in that different partitionings may be uncovered for different orderings of instances. In the presence of a strong categorical structure, the algorithm picks out the obvious categories and, as we will discuss later, there usually is little practical consequence to the different categories it extracts in the case of weak category structure. The iterative algorithm is also extremely fast. A Franz LISP implementation categorized the 290 items from Michalski and Chilausky's (1980) data set on Soybean disease (each with 36 values) in 1 CPU minute on a Vax 780 or on a MAC II. This is without any special effort to optimize the code. It also diagnosed the test set of 340 soybean instances with as much accuracy as apparently did the specially crafted system of Michalski and Chilausky (1980).

The first experiment in Medin and Schaffer (1978) is a nice one for illustrating the detailed calculations of the algorithm. They had subjects study the following six instances each with binary features:

| | |
|---|---|
| 1 1 1 1 1 | 0 0 0 0 0 |
| 1 0 1 0 1 | 0 1 0 0 0 |
| 0 1 0 1 1 | 1 0 1 1 0 |

The first four binary values were choices in visual dimensions of size, shape, color, and number. The fifth dimension reflects the category label. They then presented these 6 objects without their category label plus six new objects without a label: 0111_, 1101_, 1110_, 1000_, 0010_ and 0001_. Subjects were to predict the missing category label.

We derived simulations of this experiment by running the program across various random orderings of the stimuli and averaging the results. Figure 1 shows one simulation run where we used the order 11111, 10101, 10110, 00000, 01011, 01000 and had the coupling probability $c = 0.5$ (see Equations 4 and 5) and set all $\alpha_j = 1$ (see Equation 7). What is illustrated in Figure 1 is the search behavior of the algorithm as it considers various possible partitionings. The numbers associated with each partition are measures of how probable the new item is given the category to which it is assigned in that partition. These are the values $P(k)P(F|k)$ calculated by Equations 4 through 11. Thus, we start out with categorizing 11111 in the only possible way–that is, assigning it to its own category. The probability of this is the prior probability of a 1.0 on each dimension or $(0.5)^5 = 0.0313$. Then we consider the two ways to expand this to include 10101 and choose the categorization that has both objects in the same category because that is more likely. Each new object is incorporated by considering the possible extensions of the best partition so far. We end up choosing the partition {11111, 10101, 10110}, {00000, 01000}, {01011} which has three categories. Note the system's categorization does not respect the categorization of Medin and Schaffer.

Having come up with a particular categorization, we then tested the algorithm by presenting it with the 12 test stimuli and assessing the probabilities it would assign to the two possible values for the fifth dimension which is label. Figure 2 relates our algorithm to their data. Plotted along the abscissa are the 12 test stimuli of Medin and Schaffer in their rank order determined by subjects' confidence that the category label was a 1. The ordinate is the algorithm's probability that the missing value was a 1. Figure 2 illustrates three functions for different ranges of the coupling probability. The best rank order correlation was gotten for coupling probabilities in the range 0.2 to 0.3.

Using this coupling probability the rank order correlation was 0.87. Using a coupling probability of 0.3 rank order correlations of 0.98 and 0.78 were obtained for two slightly larger experimental sets used by

*Figure 1.* An illustration of the operation of the iterative algorithm in the material from the first experiment of Medin and Schaffer (1978).

Medin and Schaffer. These rank order correlations are as good as those obtained by Medin and Schaffer with their many-parameter model. It also does better than the ACT* simulation reported in Anderson, Kline,

*Figure 2.* Estimated probability of category 1 for the 16 test stimuli in the first experiment of Medin and Schaffer (1978). Different functions are of different ranges of the coupling probability.

and Beasley (1979). We have set the coupling probability $c$ to 0.3 throughout our applications.

The reader will note that the actual probabilities of category labels estimated by the model in Figure 2 only deviate weakly above and below 0.5. This reflects the very poor category structure of these objects. With better structured material much higher prediction probabilities are obtained.

Detailed descriptions of the application of the algorithm to particular experiments can be found in Anderson (1990) and Anderson (in press). However, we will briefly review the applications of the algorithm to a number of empirical phenomena. The following are among the empirical phenomena we have successfully simulated:

1. Extraction of Central Tendencies, Continuous Dimensions. For continuous dimensions the Bayesian model implies that categorization should vary with distance from central tendency. This enables the model to simulate the data of Posner and Keele (1968) on categorization of dot patterns and Reed (1972) on categorization of faces.
2. Extraction of Central Tendencies, Discrete Dimensions. The model implies that stimuli should be better categorized if they display the majority value for a dimension. This enabled the model to simulate the data of Hayes-Roth and Hayes-Roth (1977).
3. Effect of Individual Instances. If an instance is sufficiently different from the central tendency for its assigned category, the model will form a distinct category for it. This enables the model to account for the data of Medin and Schaffer (1978) on discrete dimensions and Nosofsky (1988) on continuous dimensions.
4. Linearly Separable vs. Non-Linearly Separable Categories. In contrast to some categorization models, this model is able to learn categories that cannot be separated by a plane in a $n$-dimensional hyperspace. This is because it can form multiple internal categories to correspond to an experimenter's category. This enables the model to account for the data of Medin (1983) on discrete dimensions and Nosofsky, Clark, and Shin (1989) on continuous dimensions.
5. Basic-Level Categories The internal categories that the model extracts corresponds to what Rosch (1976) meant by basic-level categories. Thus, it can simulate the data of Murphy and Smith (1982) and Hoffman and Ziessler (1983).[1]
6. Probability Matching Faced with truly probabilistic categories and large samples of instances the model will estimate probability of features that correspond exactly to the empirical proportion. Thus, it predicts the data of Gluck and Bower (1988) on probability matching.
7. Base-Rate Effect Because of Equation 4 this model predicts that usually there will be a greater tendency to assign items to categories of large size. Thus, it handles the data of Homa and Cultice (1984). It also reproduces the more subtle interactions of Medin and Edelson (1988).
8. Correlated Features As noted earlier the model can handle categories with correlated features by breaking out separate internal categories

[1] For a similar application, see Gluck and Corter (1985).

for each feature combination. Thus, it handles the data of Medin, Altom, Edelson, and Freko (1982)

9. Effects of Feedback If the category structure of the stimuli is strong enough the model can extract the categories without any feedback as to category identity. In the face of weak category structure, it is necessary to provide category labels to get learning. Thus, this model reproduces the data of Homa and Cultice (1984)

10. Effects of Input Order In the presence of weak-category structure, the categories the model forms is sensitive to presentation order. In this way we are able to simulate the data of Anderson and Matessa (in press) and Elio and Anderson (1984).

All these simulations were done with a constant setting of the parameters: $c$ from Equations 4 and 5 at 0.3, $\alpha_j$ from Equation 7 at 1, $\lambda_0$ from Equation 9 at 1, $a_0$ from Equation 10 at 1, $\mu_0$ from Equation 11 at the mean of the stimuli, and $\sigma_0^2$ from Equation 12 at the square of 1/4 the stimulus range. All of these are plausible settings and often correspond to conventions for setting Bayesian non-informative priors.

## 4. Comparisons to Cheeseman, Kelly, Self, Stutz, Taylor, and Freeman (1988)

The Bayesian character of this classification model raises the issue of its relationship to the AUTOCLASS model of Cheeseman et al. While it is hard to know how significant the differences are, there are a number of points of contrast.

Algorithm Rather than an algorithm that iteratively incorporates instances into an existing category structure, Cheeseman et al. use a parameter searching program that tries to find the best fitting set of parameters. The Cheeseman program is quite fast and is not sensitive to the order of the examples. On the other hand, it does not easily generate predictions that can be incrementally updated with each example.

Number of Classes AUTOCLASS has a bias in favor of fewer classes whereas this bias is setable in the rational model according to the parameter $c$. AUTOCLASS does not calculate a prior corresponding to the probabilities of various partitionings.

Conditional Probabilities It appears that AUTOCLASS uses the same Bayesian model as we do for discrete dimensions. The treatment of continuous dimensions is somewhat different although we cannot discern its exact mathematical basis. The posterior distribution is a normal distribution which will only be slightly different than the $t$-distribution we use. Both AUTOCLASS and the rational model assume the various distributions are independent.

Qualitatively, the most striking difference is that AUTOCLASS derives a probability of an object belonging to a class whereas the rational model assigns the object to a specific class. However, Cheeseman et al. report that in the case of strong category structure the probability is very high that the object comes from a single category.

## 5. Order Sensitivity

The categorization algorithm that we have described is order sensitive and this has been a point of criticism of the model (Ahn & Medin, 1989). If critical examples have accidental similarities the model will create pseudo-categories around these. If the initial examples have exaggerated differences the algorithm will fail to identify the true categories but split them into lower-level categories. The basic problem of the algorithm is that it is unable to split categories that it has formed into subcategories or to merge existing categories into larger categories. In Anderson (1990) we showed subjects do display some sensitivity to order but much less than our algorithm.

An interesting question concerns the consequences of this order sensitivity from the goal of the rational analysis which to maximize prediction where maximal prediction is defined with respect to the ideal algorithm (Equation 1). It is usually impossible to calculate the predictions of the ideal algorithm but the first experiment of Medin and Schaffer (1978 - see Figures 1 and 2) used a sufficiently small stimulus set that this is feasible. We calculated the ideal probabilities for the test stimuli in Figure 2 using $c = 0.5$ and $\alpha_j = 1$. At $c = 0.5$, depending on ordering, the iterative algorithm selects out one of the following three partitionings:

A: (01011) (00000, 01000) (10101, 10110, 11111)

B: (11111, 01011) (00000, 01000) (10101, 10110)

C: (10101, 10110, 11111) (01011, 00000, 01000)

A is the partitioning illustrated in Figure 1; it has log probability $-25.77$;[2] it occurs 22% of the time. B has log probability $-26.52$ and occurs 16% of the time. C has log probability $-25.64$ and occurs 61% of the time. C is the most probable partitioning of all. By comparison a partitioning that merges all into one category has log probability $-26.50$, one that breaks them up into single categories has log probability $-27.37$,[3] and something awful like (11111, 00000) (01000, 10101), and (10110, 01011) has log probability $-32.07$. The median probability of the 203 partitions expressed in log terms as $-28.66$ or about 5% the probability of the most probable.

The Medin and Schaffer stimulus set has weak category structure and the algorithm does not always find the most probable partition. In the case of strong category structure the program extracts the most probable interpretations independent of order of presentation. Fisher (1987) reports a similar result for his Cobweb program.

The critical issue is how well the various partitions do at predicting features. Therefore, we looked at various partitions with respect to predicting the fifth dimension of the 12 stimuli illustration in Figure 2. We looked at the correlations among the predictions of various procedures. Table 1 reproduces the correlation matrices among the predictions of the three partitionings A, B, and C, their weighted average (as produced by the iterative algorithm), and the weighted prediction from the ideal algorithm (Equation 1). As can be seen, they all are relatively highly correlated with the ideal and, except for A and C, with each other. The weighted average of A through C is very highly correlated ($r = .96$) with the ideal. This suggests that there is relatively little cost associated with using the iterative algorithm.

[2] What we are calculating is the product of $P(k|F)$ (from Equation 3) for all the instances. This represents the likelihood of the data given the parameters $c$ and $\alpha$.

[3] A singleton category structure is less likely than a single category because of the high value of $c$. At $c = .3$, the log probability of the single category becomes $-28.11$ and the log probability of the singleton categories is $-22.95$.

*Table 1.* Correlation matrix among various algorithms with respect to predicting the stimuli in Figure 2.

| | IDEAL | A | B | C |
|---|---|---|---|---|
| PARTITION A | 0.89 | X | X | X |
| PARTITION B | 0.98 | 0.86 | X | X |
| PARTITION C | 0.80 | 0.49 | 0.81 | X |
| AVERAGE | 0.96 | 0.78 | 0.96 | 0.92 |

It is interesting that the prediction is not particularly good using the most probable partition. This reflects that the most probable partition has only 5% of the probability of the 203 possible partitions of the 6 stimuli. As the set size gets larger or as the category structure improves, the most probable partition will tend to dominate the prediction. This suggests that it makes sense for the system to strive for the most probable partition but prediction from some other highly probable partition may be as good or better.

### 5.1 A Hierarchical Algorithm

Considerations of the order-sensitivity of the algorithm has led us to consider other incremental algorithms that are less order sensitive. We were also interested in producing a hierarchical category structure and exploring the issue of whether other levels in the hierarchy, besides the basic level, might be useful for prediction.

We have developed another algorithm which is somewhat more successful at identifying the maximally probable partition but avoids considering all possible partitions as does the ideal algorithm. This algorithm organizes the data into a hierarchical structure. Figure 3 illustrates a hierarchical structure to organize the stimuli from the first experiment of Medin and Schaffer. Having built such a hierarchical representation of the stimulus set, our algorithm tries to determine which partitioning within the hierarchy offers the optimal decomposition of the stimulus set. This will depend on the setting of the coupling parameter $c$. The higher the value of $c$ the larger the categories that the algorithm will tend to select. Given the structure in Figure 3, the algorithm will select the top-level node as the single category for values of $c$ greater than 0.7. For values of $c$ in the range 0.39 to 0.70 it will

*Figure 3.* Hierarchical organization of stimuli from first experiment of Medin and Schaffer (1978).

select the two subnodes. For values of $c$ from 0.32 to 0.39 it will select the bottom-level nodes except for 00000 and 01000 which it will merge into a single category. Below 0.32 it selects singleton categories.

The basic algorithm for growing this network is as follows:

1. At any point in time, it will have a hierarchical organization for the seen instances and, given a value of $c$, it will have identified a set of categories.
2. As before, given a new instance, it will determine a category to associate with this instance.
3. If that is an existing category it will sort the new instance to a location below that category node.
4. If it is a new category it will sort that category to some location in the hierarchical structure that exists above the category nodes.
5. It will search upward from where the new item was inserted to see if some change in of the category structure is warranted. Note this

does not reorganize the hierarchy but only changes which nodes in the hierarchy might be considered category nodes.

Figure 4 illustrates the basic logic for sorting and inserting a new instance into the hierarchical structure. We have an existing hierarchical structure consisting of a node a with subnodes b and c. We have a new instance d that we want to incorporate somewhere in the hierarchical structure under a. There are three possibilities: (i) d will be associated with the hierarchical structure dominated by b; (ii) d will be associated with the hierarchical structure dominated by c; or (iii) a binary branch will be created with b and c in one and d in the other. The way to choose among these is to identify the branching that will yield the maximally probable pair of categories to cover all the items under b, c, and d. For (i) we consider the product of the probability of the category consisting of b + d and the category consisting of c. For (ii) we consider the b category and the c + d category. For (iii) we consider the b + c category and the d category.

It is worthwhile comparing the performance of this hierarchical algorithm with that of the previous algorithm. We used the material of Medin and Schaffer for this purpose. There are 720 possible orderings of the 6 stimuli. With $c = 0.5$, the old algorithm identified 3 different categories and identified the optimal categorical structure 61% of the time, and produced categorical structures with average log probability of $-25.81$. The hierarchical algorithm identified 6 different categories, but identified the optimal category structure 80% of the time, and produced categorical structures with average log probability of $-25.74$. Thus, there is not much difference in average goodness, but the hierarchical algorithm is somewhat more successful at finding the optimal structure. With larger stimulus sets, we have not been able to be exhaustive but the hierarchical algorithm does appear more stable and does more often select the ideal structure.

We have also explored hierarchical algorithms that are not order sensitive. Basically, they used classical clustering techniques (Annenberg, 1973) to create hierarchies of stimuli sorted according to similarity where similarity is measured by Equations 6–8. Such algorithms are more expensive computationally because they must perform all pairwise comparisons. We have not found that they yield notably better results.

A good case for illustrating the problems of these algorithms is the iris data base of Fisher (1936). According to Fisher there are three under-

*Figure 4.* Choose structure to maximize probability. Should it depend on prior probability?

lying types of Irises. Our algorithms, whether hierarchical or not, and if hierarchical , whether iterative or not, fail to identify this category structure. They always identify one of the categories, Iris Setosa, but either fail to separate the other two (Iris Versicolor and Iris Virginica) or split them up inappropriately (as defined by Fisher). Cheeseman et al. claim their algorithm to be successful but our observation is that it also produces an inappropriate splitting into three categories that do not correspond to the original three. We also have observed of human subjects that they also extract two categories or produce an inappropriate splitting into three. This notwithstanding, it can be shown that the original categorization produced by Fisher is more probable than the two-category solution or the various inappropriate three-category solutions. However, it is apparently impossible to find this more probable partitioning given the various approaches, artificial or human. However, it turns out that prediction of features is not enhanced by the more probable partitioning. Thus, it is not clear that we should consider the behavior of these various algorithms as failures.

### 5.2 Genus and Instance Level Identification

While it does not seem that the hierarchical approach produces substantially better categorical organization, we think that there might be some significance to the levels of our hierarchy. There are at least two other levels of the hierarchy which are significant for purposes of prediction. At a higher level is the genus[4] and at the lower level is the individual. We will discuss the significance of prediction at each level.

The genus level offers a level of aggregation above the species. A genus corresponds to a group of biologically related species which are more similar to one another than are arbitrary pairs of species. The significance of the genus level does not come in making predictions about known properties of known species. For instance, we are much better off predicting the cat-chasing propensity of Fido knowing that he is a dog than knowing he is a mammal. The significance of the genus level comes in making predictions about unknown properties of a known species (e.g., whether Fido has a spleen) and making predictions about unknown species.

In Bayesian terms, the significance of the genus level is that it can be used to set more informed priors for the species under the genus. This will help in making predictions about new species and about unexperienced properties of existing species. The interesting complication is that these priors themselves depend on estimates of the parameters for the existing species which in turn depend on the priors. Thus, it might seem that we have a difficult joint estimation problem. The typical Bayesian approaches to such estimation problems are what are called hierarchical methods (Berger, 1985, Section 4.6). The technical development of such

[4] Our use of the term genus is in its more general sense to refer to a kind and does not imply the precision that is involved in the distinction among genus, family, order, class, and phylum in biology. We suspect that the level useful in prediction might be considerably above the biological genus level and actually closer to the phylum level.

methods can be quite complex and is not justified here since data has not yet been gathered that requires such complex quantitative analysis. We will simply note, for our purposes, they provide a rationale for making estimates of the mean and variance within a species sensitive to estimates mean and variances for other species within a genus.

There certainly is evidence that people have this sensitivity. Even young children have expectations about the properties of new animals based on animals which they have seen (Carey, 1985). They also have expectations that certain dimensions are less variable for certain types of categories. Thus, there is the expectation that animals within a category will have the same constitution while artifacts within a category will have the same function (Gelman, 1988). Moreover, these expectations show developmental trends to more accurate forms as experience accumulates.

The experiment of Nisbett, Krantz, Jepson, and Kunda (1983) also illustrates differential sensitivity to variance in categories of different kinds. They asked subjects to suppose they had a sample of a new mineral, a new bird, or a new tribe of people from a new island. They were given samples of different size and told that all the objects within the sample had some property. Subjects were willing to extrapolate from a single observation for some dimensions like conductivity of the mineral or color of the tribe of people whereas they required 20 observations before they were able to extrapolate with any confidence for other dimensions like the obesity of the people.

This ability to show sensitivity to variance is one thing that distinguishes this hierarchical Bayesian approach to categorization from most others. Many approaches (e.g., instance-based models) would predict that subjects would be biased in their estimate of the mean of a new species by the mean of existing species. These other approaches do not have the mechanisms, however, for showing a similar sensitivity to variance.

### 5.3 Individual-Level Identification

The individual provides a much lower level of aggregation below the category. For purposes of prediction, there is a real advantage to identifying a repetition of an individual and making predictions from the individual rather than the category. This is because the individual may reliably deviate from the mean of the category and because many features are much more certain at the individual level than at the category.

Retrieving an individual and making a prediction on this basis corresponds to a memory retrieval. From this perspective the difference between memory and categorization concerns whether prediction is being made at the individual level or the category level. It is basically the same logic of prediction; however, it needs to be parameterized differently:

(a) To reflect the fact that individuals repeat themselves much less often than categories, we use a lower value of $c$.

(b) We need to capture the fact that the features are much less likely to change. To accommodate this we need to have lower values of the $\alpha_i$ for discrete dimensions and much smaller values of $\sigma_0^2$ for continuous.

There has been a lot of speculation as to how categorization behavior relates to memory behavior. The instance-based models (Medin and Schaffer, 1978, Nosofsky, 1986) would argue that everything is really memory-based while connectionist models (McClelland, Rumelhart, and Hinton, 1986) would argue that there are no separate representations of instances and everything is merged together. They both try to account for differences between categorization and memory by arguing that a single representation is differently processed. This model offers a representation which distinguishes the two levels but uses the same Bayesian logic at both levels. Of course, the rational representation is only an acknowledgment of the fact that there are individuals and categories in the real world. It does not really make any claims about how they are processed in the head. Anderson (in press) can be consulted for experimental evidence that people do make different predictions when they operate at the category level or the individual level.

## 6. Summary

In summary, we have identified an iterative Bayesian algorithm which is fast, yields near-optimal predictions about stimulus dimensions, and which corresponds with uncanny accuracy to the behavior to humans. We have explored the potential of some hierarchical variations of the algorithm. They produce marginal improvements at best in the prediction behavior. However, there is reason to suppose that human categorization behavior has sensitivities to levels above and below the basic level category.

Our interest in the original iterative algorithm began as a way to get approximations to the ideal, computationally impossible, prediction (specified by Equation 1), it was not intended as a serious model for either human cognition nor as an AI application. However, after more than two years of exploration, we have failed to find a real improvement and continue to be surprised at how well it does.

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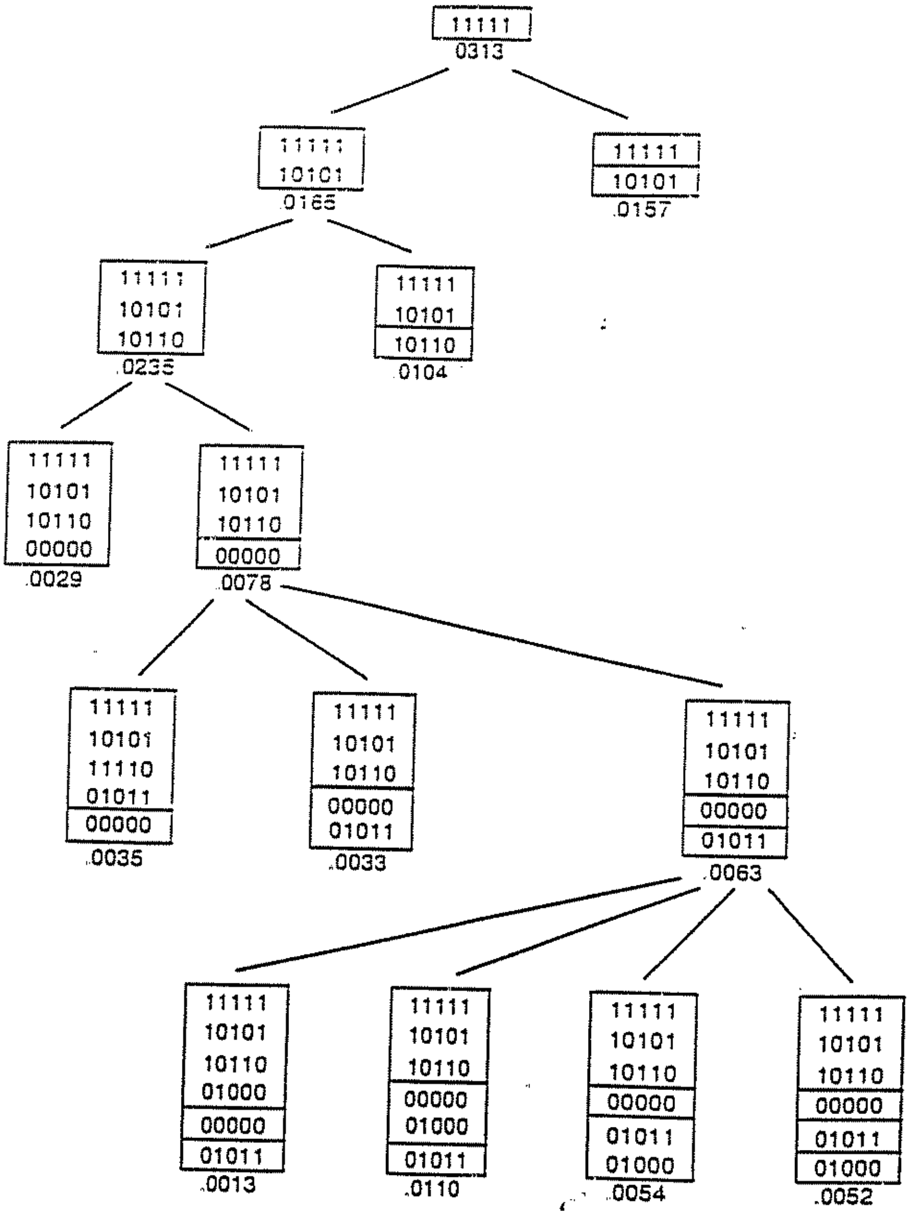


Figure 1 (reduced)

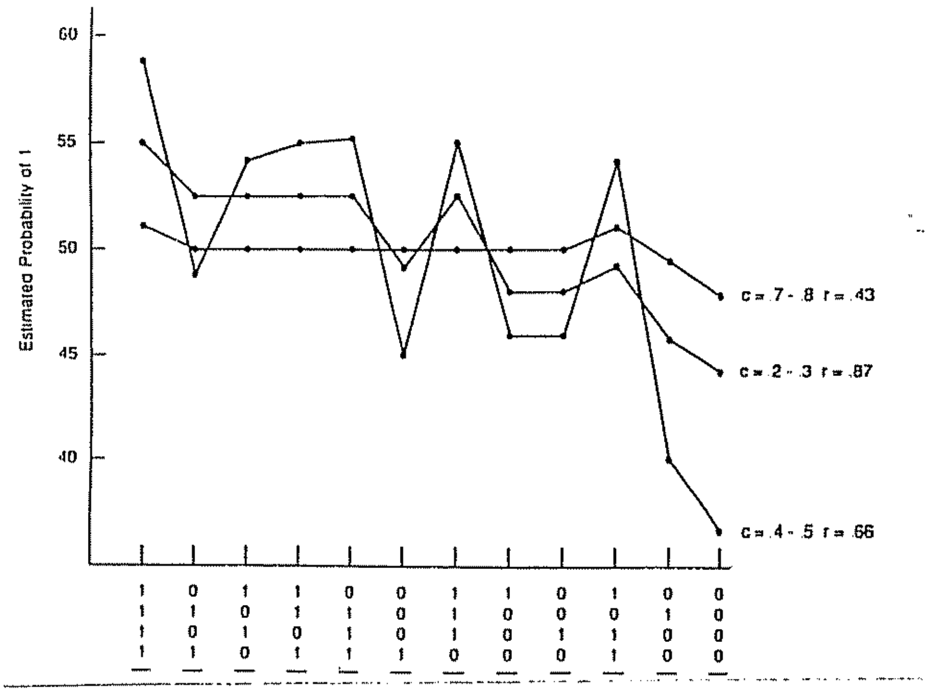


FIGURE 2 (reduced)

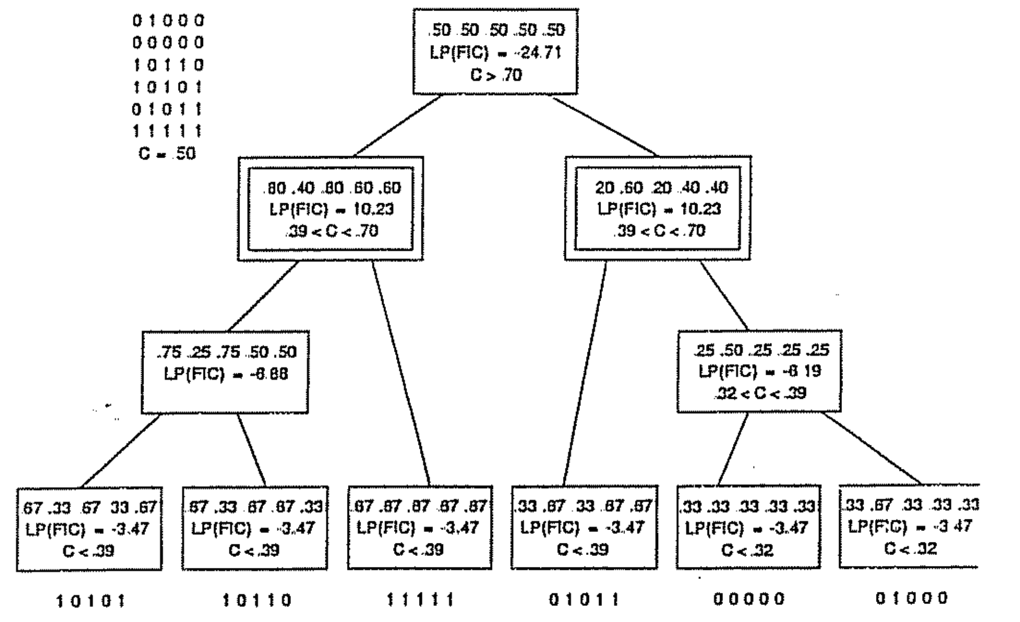


FIGURE 3 (reduced)

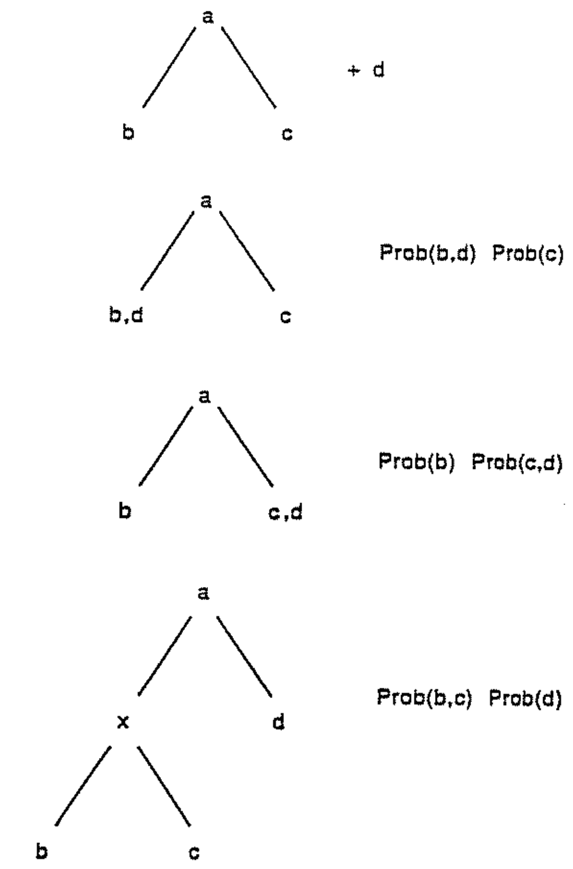
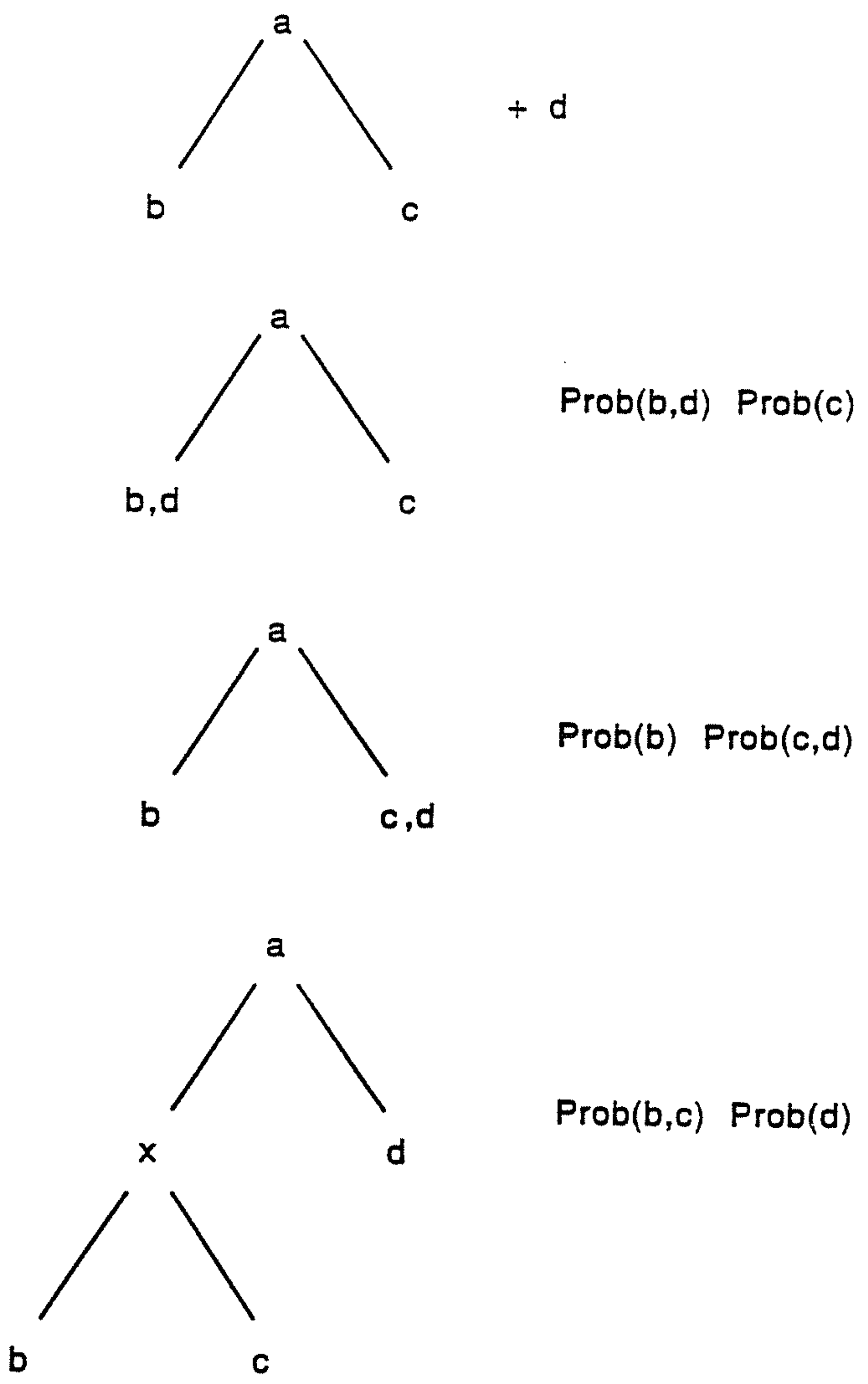


FIGURE 4 (reduced)

## FIGURE 5



Choose Structure to Maximize Probability

Should it depend on Prior Probability?

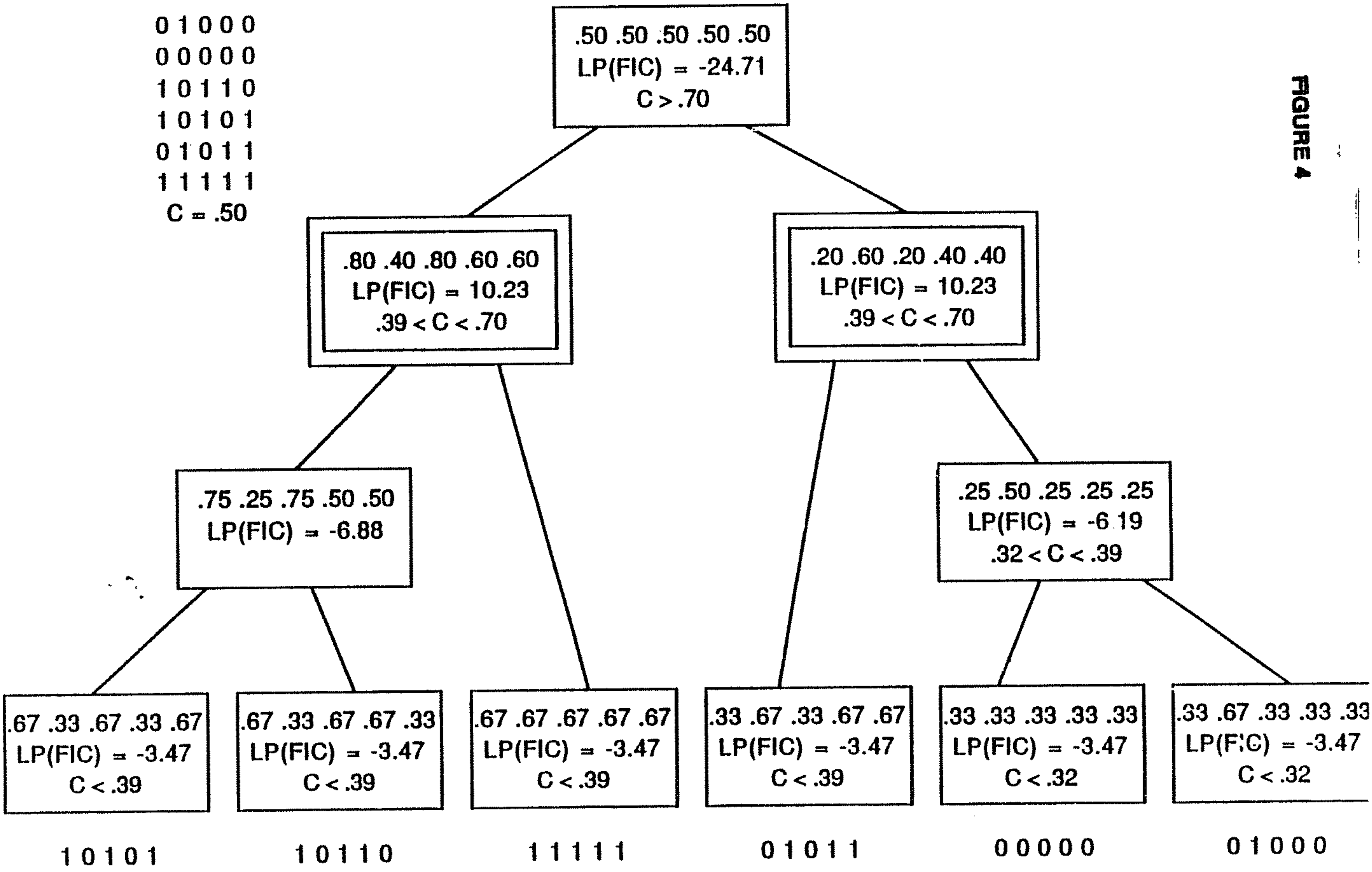

0 1 0 0 0
0 0 0 0 0
1 0 1 1 0
1 0 1 0 1
0 1 0 1 1
1 1 1 1 1
C = .50
.50 .50 .50 .50 .50
LP(FIC) = -24.71
C > .70
.80 .40 .80 .60 .60
LP(FIC) = 10.23
.39 < C < .70
.20 .60 .20 .40 .40
LP(FIC) = 10.23
.39 < C < .70
.75 .25 .75 .50 .50
LP(FIC) = -6.88
.25 .50 .25 .25 .25
LP(FIC) = -6.19
.32 < C < .39
.67 .33 .67 .33 .67
LP(FIC) = -3.47
C < .39
.67 .33 .67 .67 .33
LP(FIC) = -3.47
C < .39
.67 .67 .67 .67 .67
LP(FIC) = -3.47
C < .39
.33 .67 .33 .67 .67
LP(FIC) = -3.47
C < .39
.33 .33 .33 .33 .33
LP(FIC) = -3.47
C < .32
.33 .67 .33 .33 .33
LP(FIC) = -3.47
C < .32
1 0 1 0 1
1 0 1 1 0
1 1 1 1 1
0 1 0 1 1
0 0 0 0 0
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FIGURE 4

FIGURE 2

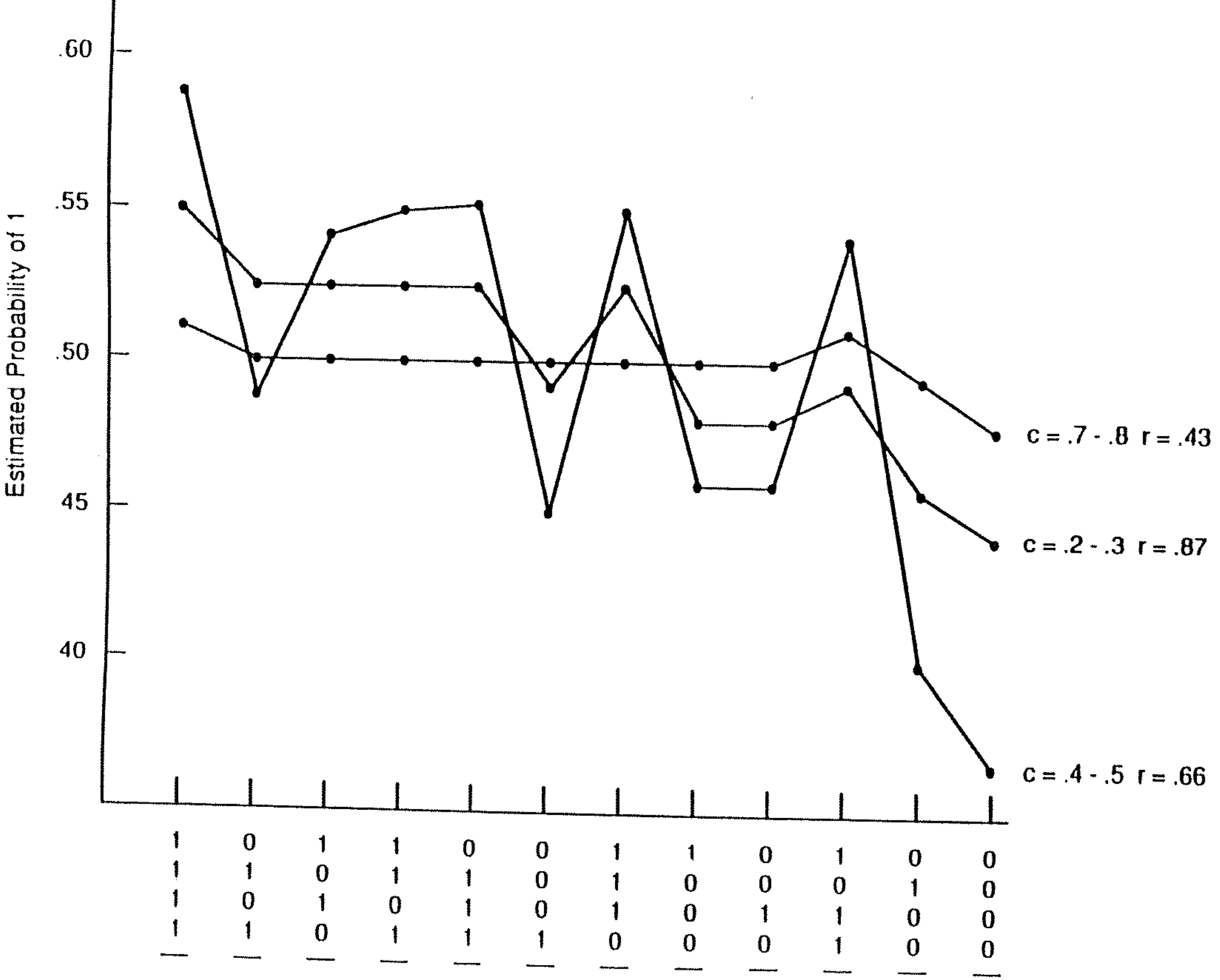

Figure 3.2 Estimated probability of category 1 for the 16 test stimuli in the first experiment of Medin & Schaffer (1978). Different functions are for different ranges of the coupling probability.

## FIGURE 1

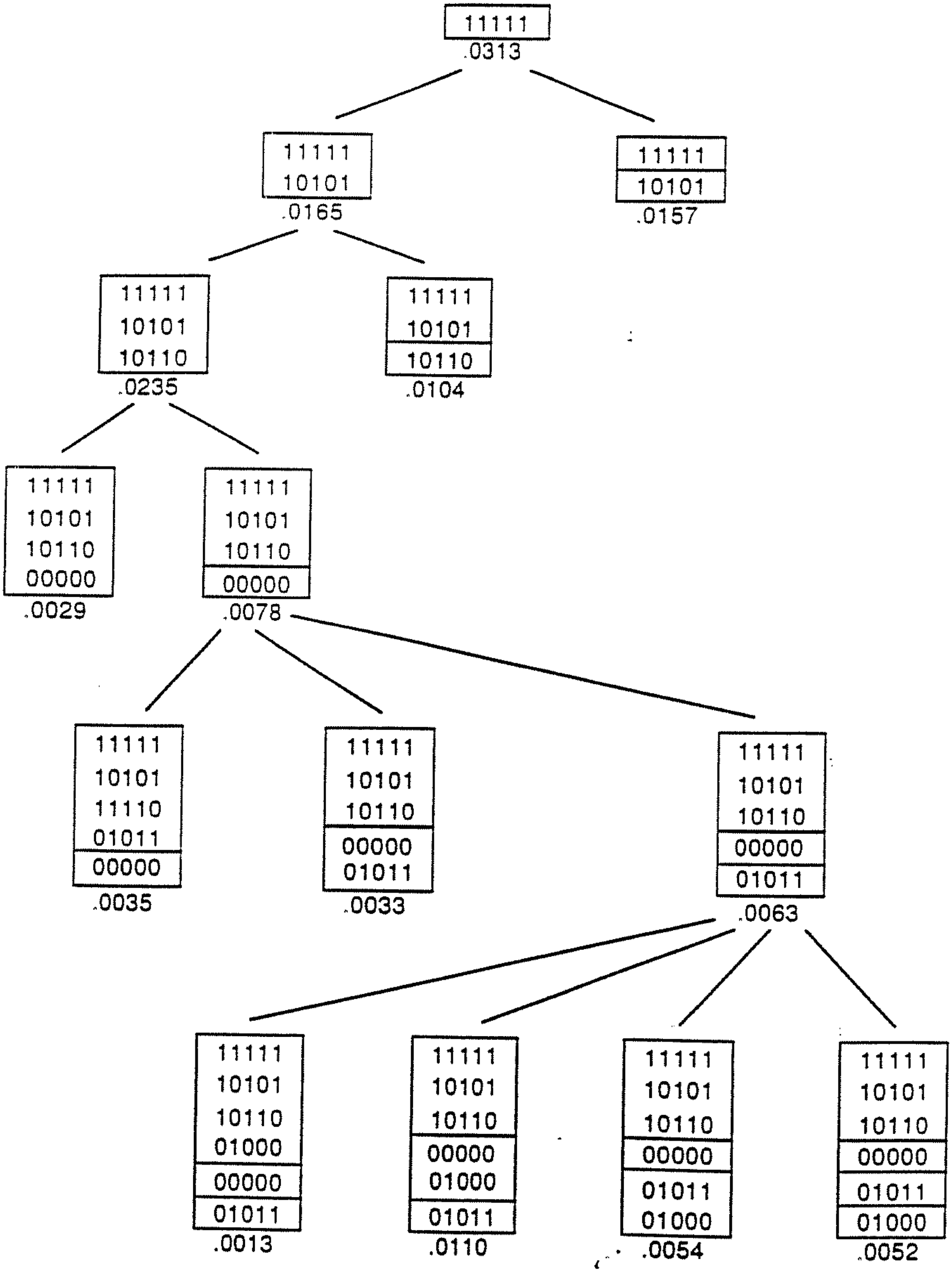

Figure 3.1 An illustration of the operation of the iterative algorithm in the material from the first experiment of Medin & Schaffer (1978).

FIGURE 3

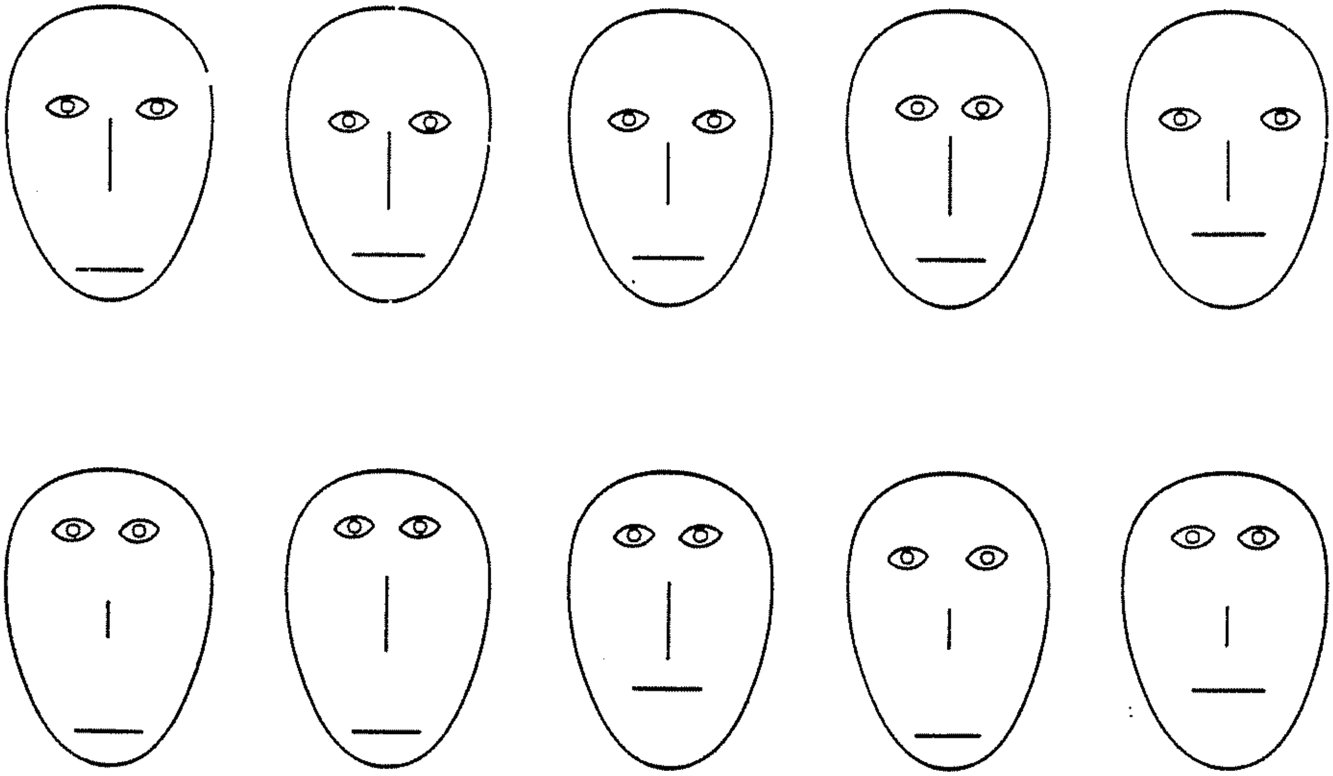

**Figure** *The faces in the two artificial categories in Reed's experiment (1970) studying schema abstraction with respect to faces. The faces in the top row are from category 1, and the faces in the bottom row are from category 2. (From Reed, S. K Pattern recognition and categorization.* Cognitive Psychology, *1972, 3, 382-407.)*