Probabilistic Production Selection in Act-R

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ABSTRACT

In ACT-R, some subsymbolic mechanisms can be switched on that transform ACT-R into a stochastic theory using random variables to generate choice probabilities. This paper asks for the relation between probabilities of production selection and the corresponding random variables. Starting with the assumption that choice probabilities governing production selection satisfy the requirements of Luce's choice axiom, it is shown that these probabilities can be generated by a system of independent and identically double-exponentially distributed random variables.

Keywords

Production selection, choice probabilities, choice axiom, double-exponential distribution, logistic distribution, scale types.

INTRODUCTION

In ACT-R models, the system selects productions out of the set of all productions with a first pattern in the left-hand side matched by the current goal. In the deterministic version, that production is selected for which the '*PG–C*'-value is maximal and larger zero. '*PG–C*' is sometimes called the "evaluation" of production *i* and represented by E_i. In the probabilistic version, a random variable ε is added to E_i and the production with the largest positive realization of the random variable 'E_i+ε' is chosen. Anderson and Lebiere (1998, p. 64) assume independent and logistically distributed ε-components with distribution function

(1) $$F(\varepsilon) = \frac{1}{1 + e^{-\varepsilon/s}}.$$

In 'Conflict Resolution Equation 3.4' Anderson and Lebiere (1998, p. 65) claim that the resulting choice probabilities are

(2) $$P(i) = \text{Prob}\{\text{selection of production } i\} = \frac{e^{E_i/t}}{\sum_j e^{E_j/t}}$$

with $t = s\sqrt{2}$. Unfortunately, equation (2) does not follow from (1) (see Yellott, 1977). It gives a good approximation, but approximations sometimes have properties different from those of what is approximated.

This paper starts with assumptions that lead to equation (2) and tries to find additive random variables, which can be justified by these assumptions. The reason why such an upside-down procedure seems promising is that (2) conforms to Luce's (1959) "choice axiom".

STOCHASTIC THEORY OF CHOICE

Choice Probabilities

Equation (2) is not precisely formulated. Obviously, the probability of choosing or selecting something depends on the set of alternatives from which to choose. Therefore, choice probabilities are the function of *two* arguments, an option and a set of options. Given option set *B* and option $a \in B$, we write *P*(*a*, *B*) to denote the probability of choosing *a* if *B* is the set of feasible options. A triple $\langle A, M, P \rangle$ is a *structure of choice probabilities* if an only if

- *A* is a set,
- *M* is a nonempty subset of 2^A whose members are nonempty and finite,
- *P* is a real-valued function with domain $\{(a, B) \mid (a \in B \in M)\}$ such that

$$P(a, B) \geq 0 \quad \text{and} \quad \sum_{b \in B} P(b, B) = 1$$

(Suppes at al., 1989, p. 384). Such structures are called *finite* if and only if *A* is finite, and they are *closed* if and only if *A* is finite and $M = \{B \subseteq A \mid B \neq \emptyset\}$.

Often, the notion of choice probabilities is extended to cover values for subsets of options. For $B \subseteq A$, we write

(3) $$P(B, A) = \sum_{b \in B} P(b, A)$$

to refer to the probability of choosing any element of *B* out of option set *A*.

In case of stochastic production selection in ACT-R, we have a structure of choice probabilities that is finite but not closed. Furthermore, we have to take into account that the sum of probabilities over all available options equals 1 only if each option set contains the 'no-selection option' which is chosen if the maximum of E_i+ε is negative and, therefore, no production at all is selected. We use the symbol o to denote this null-option. One of the problems with equation (2) is that it does not take into account this 'no-selection option'.

In the literature on theories and models of choice behavior, there is a distinction between 'constant representation' and 'random variable representation' models. *Constant representation models* specify a real-valued function ϕ on *A* such that for each possible size β of a finite option set there exists a real-valued function F_β in β real arguments such that

(4) $$P(a,B) = F_\beta(\phi(a), \phi(b), \quad \phi(h))$$

with each F_β strictly increasing in its first argument and strictly decreasing in the other arguments (Suppes et al., 1989, p. 410). In a *random variable representation model*, there exists a collection $\mathbf{U} = \{\mathbf{U}_a \mid a \quad A\}$ of random variables such that

(5) $$P(a,B) = \mathrm{Prob}\{\mathbf{U}_a = \max\{\mathbf{U}_b \mid b \quad B\}\}.$$

While constant models simply impose plausible constraints on choice probabilities, random variable models postulate a covered process representing options by random variables that are compared to reach a final choice. Since Thurstone (1927), such processes are called "discriminal processes". Equation (1) belongs to a random variable model with $\mathbf{U}_i = E_i + \varepsilon$ and equation (2) to a constant representaion model with $\phi(i) = e^{E_i/t}$.

The 'Choice Axiom'

Luce's (1959) choice axiom, which is equivalent to the 'strict utility model', is a good candidate for a constant representation of stochastic production selection in ACT-R. Its basic assumptions are rather simple but not trivial, and a lot of important consequences are well known. The special case of option sets with two options only was originally proposed by Bradley and Terry (1952) and Bradley (1954a, 1954b, 1955). Therefore, another name for the same theory is ‚Bradley-Terry-Luce system' or simply ‚BTL scale' (Suppes & Zinnes, 1963; Luce & Galanter, 1963). Modeling confusion probabilities in speech perception, Clarke (1957) stated what is in principle the same assumption.

There are several equivalent versions of the choice axiom. They all use the extension of the choice-probability concept given in equation (3). One of them additionally introduces *conditional choice probabilities*

(6) $$P((D \mid C), B) = \frac{P(D \cap C, B)}{P(C,B)}$$

provided $P(C,B) \neq 0$. The axiom itself is stated as

Choice Axiom 1: A set of choice probabilities defined for all the subsets of a finite set *A* satisfies the choice axiom provided that for all *a*, B, *C* such that $a \quad C \quad B \quad A$

(7) $$P(x,C) = P((x \mid C), B),$$

whenever the conditional probability exists (Luce & Suppes, 1965, p. 336).

Another version avoids the introduction of conditional choice probabilities and states the axiom as follows:

Choice Axiom 2: A closed structure of choice probabilities, with $P(B,A) \neq 0$ for all $B \quad A$, satisfies the choice axiom iff for all $C \quad B \quad A$

(8) $$P(C,A) = P(C,B)P(B,A)$$

(Suppes et al., 1989, p.416).

The basic principle is that the process leading to the selection of any element in *C* given option set *A* can be thought of as a two(or more)-step process consisting of a choice of *C* from *B* and a choice of *B* from *A*, that these two choices are independent, and that the process leads to the same result independent of which one of the possible *B*'s is taken for the intermediate state. According to the choice axiom, a selection of an option *a* from a set *C* is just that, regardless whether *C* is the given option set or whether there is a larger option set B and we only look for choices of *a* at those occasions on which an element of C was chosen from B.

The choice axiom is equivalent to the ‚constant ratio rule' (Clarke, 1957), which states that for all options $a,b \quad B \cap C$ and $B,C \quad M$ holds

(9) $$\frac{P(a,B)}{P(b,B)} = \frac{P(a,C)}{P(b,C)}$$

provided both denominators are not zero. The ratio of the choice probabilities of two options is the same for all option sets containing both of them. Something like a ‚strength of preference' of *a* over *b* is taken as independent of what else is available as an option. This rule can be viewed as a probabilistic brother of the principle of 'independence of irrelevant alternatives' which is prominent in deterministic choice theory.

Furthermore, the choice axiom is equivalent to the 'strict utility model' that assumes the existence of a function $v : A \rightarrow \quad$ such that

(10) $$P(a,B) = \frac{v(a)}{\sum_{b \quad B} v(b)}.$$

This is the format of Anderson and Lebiere's (1998, p. 65) 'conflict resolution equation 3.4'.

The choice axiom may look simple, but it is by no means trivial. Imagine little Alice who has to decide whether she prefers a pony or a bicycle as birthday present from Uncle John. Poor Alice is totally indifferent between these two options. Therefore, we can assume

$$v(\text{pony}) = v(\text{bicycle}),$$

which leads directly to a choice probability of 1/2 for each of the two options. Enlarging the option set by adding a perfect copy of the bicycle as a third alternative would — under choice-axiom assumptions — reduce the probability of voting for the pony from 1/2 to 1/3. In real life, this does not make much sense. The example shows that the choice axiom presupposes a strict form of distinctness of the feasible options. Fortunately, this creates no special problems for productions in Act-R. It does not make much sense to write a model with two or more identical copies of one production. New productions created by compiling dependencies are never perfect copies of each other.

PRODUCTION SELECTION

When formulating the assumption that the choice axiom holds for stochastic production selection in ACT-R, we have to take into account the already mentioned requirement that every option set must contain the no-selection option o. Given a current goal *g*, there is a set of productions *R*(*g*) with a first pattern matching *g*. The option set from which the system can choose consists of all productions from *R*(*g*) plus the no-selection option o. We formulate the basic assumption by using the strict-utility form.

Selection Assumption:

Given a current goal *g*,

the set *R*(*g*) of all productions with a goal pattern matching *g*,

the 'no-selection option' o,

there exists a function $\phi : \left(R(g) \cup \{o\}\right) \rightarrow$ such that for all $i \in R(g)$

(11) $$P(i,g) = \frac{\phi(i)}{\phi(o) + \sum_{j \in R(g)} \phi(j)}$$

is the probability of production *i* being chosen if *g* is the current goal, and

(12) $$P(o,g) = \frac{\phi(o)}{\phi(o) + \sum_{j \in R(g)} \phi(j)}$$

is the probability that no production is selected and *g* is popped with failure.

The task is to find a *random-variable representation* using the same option sets and resulting in the same choice probabilities. This representation should be of the following kind.

Random Variables assumption:

Given a current goal *g*,

the set *R*(*g*) of all productions with a goal pattern matching g,

the 'no-selection option' o,

there is a function $\gamma : \left\{R(g) \cup \{o\}\right\} \rightarrow$ and a set of independent and identically distributed random variables $\mathbf{X} = \left\{\mathbf{X}(i) \mid i \in R(g) \cup \{o\}\right\}$ such that

(13) $$P(i,g) = \mathrm{Prob}\left\{ \gamma(i) + \mathbf{X}(i) = \max\left\{\gamma(j) + \mathbf{X}(j) \mid j \in R(g) \cup \{o\}\right\} \right\}$$

(14)

$$P(o,g) = \mathrm{Prob}\left\{ \gamma(o) + \mathbf{X}(o) = \max\left\{\gamma(j) + \mathbf{X}(j) \mid j \in R(g) \cup \{o\}\right\} \right\}$$

This kind of random-variable representation differs in one important aspect from Anderson's and Lebiere's (1998, p. 64) original assumption. They propose to select that production for which $\gamma+\mathbf{X}$ is maximal and larger zero. In other words, they look for the maximum in the set

$$\left\{0, \gamma(1) + \mathbf{X}(1), \gamma(2) + \mathbf{X}(2), \ldots\right\}$$

while the proposal here is to search the maximum in

$$\left\{\gamma(o) + \mathbf{X}(o), \gamma(1) + \mathbf{X}(1), \gamma(2) + \mathbf{X}(2), \ldots\right\}.$$

They take the no-selection option into account in their random-variable representation, but they do not consider it in the constant representation given in their 'Conflict Resolution Equation' (Anderson and Lebiere, 1998, p. 65). Since the option sets differ, there can be no correspondence between their two representations.

There are several possibilities to remove this inconsistency. 'Conflict Resolution equation' could be interpreted as an expression of the choice probabilities of those productions for which the realized value of the corresponding random variable is larger zero. Formally, the summation in Equation 3.4 (in Anderson & Lebiere, 1998, p.64) can be taken as going over all

(15) $$i \in \left\{R(g) \mid \left(E_i + \varepsilon\right) > 0\right\}.$$

Unfortunately, multiplying the resulting choice probabilities in such a restricted option set with the probability that there is a selection (and no 'pop the goal with failure') does generally not give the choice probability in the unrestricted set. The reason is that the 'no selection' event is not guaranteed to be independent of which production has gained the highest value. Additionally, restricting the choice set in the random model accordingly would require the introduction of truncated random variables with

(16) $$F\left(\varepsilon \leq -E_i\right) = 0,$$

which are — due to their dependence on E_i —no longer identically distributed. Such complications can be avoided by the much simpler solution proposed here.

The assumption in this paper is that all alternatives in the option set have the same status. Not only the values attached to productions are randomly fluctuating but the lower limit of productions' selectability is randomly fluctuating too.

The random-constant connection

It is an old idea that that constant and random-variable representations of choice probabilities differ only at the surface and are in principle based on the same assumptions about properties of probabilistic choice behavior. Early research in this area has concentrated on the special case of pair comparisons. "Pair comparison" refers to choices from option sets of size two. In this case, the probability $P\left(a,\left\{a,b\right\}\right)$ can be interpreted as the probability of preferring *a* over *b* and is — therefore — often formalized as $P(a \succ b)$. When Luce (1959) presented his choice axiom, he pointed out that in the special case of pair comparisons his axiom is equivalent to a random variable model in which the differences

$$\left(\left(\gamma(i) + \mathbf{X}(i)\right) - \left(\gamma(j) + \mathbf{X}(j)\right)\right)$$

are logistically distributed random variables. The question whether there are random-variable distributions leading to the choice axiom for any size of option sets, which is the question we are pursuing here, was referred to by Luce (1959, p. 144) as 'open problem B-2'.

At first sight, the relation between random variables and choice probabilities looks rather simple. Adding a constant to a random variable results in a random variable. So, we can —for simplicity— introduce for any current goal *g* the set

(17) $$\mathbf{Y} = \left\{\mathbf{Y}(i) = \mathbf{X}(i) + \gamma(i)\right\}.$$

Option *i* is chosen if $\mathbf{Y}(i)$ takes any value and all $\mathbf{Y}(j)$ take values which are not larger. Assume that for every $\mathbf{Y}(i)$ there exists a distribution function F_i and a density f_i. Then, thanks to the independence between the $\mathbf{Y}(i)$,

(18) $$P(i,g) = \int_{-\infty}^{+\infty} f_i(\xi) \prod_{j \in R(g) \cup \{o\}} F_j(\xi)\, d\xi .$$

The task is to find a family of distributions for the random variables such that this integral exists in closed form and results in an expression equivalent to equation (10).

The 'double exponential'

As already mentioned, Luce (1959) has shown that logistically distributed differences between the $\mathbf{X}(i)$ yield a system equivalent to the choice axiom for pair comparisons. Adams and Messick (1957) could prove that logistically distributed differences are even necessary for this equivalence. Therefore, the distributions of the $\mathbf{X}(i)$'s themselves must belong to a family from which logistically distributed differences can be derived. Such a family, among others, is the class of 'double exponential distributions'. Holman and Marley assumed double-exponentially distributed random variables in the general case (including larger option sets) and derived, based on equation (18), that this is sufficient for a strict utility model. There seems to exist no special publication of Holman and Marley's proof; it is referred to in Luce and Suppes (1965, p. 338), and the derivation itself is presented by Suppes et al. (1989, p. 424 f.). Finally, Yellott (1977) has shown that for closed structures of choice probabilities with at least three options the double exponential is the only distribution with the required property.

The distribution function of the double exponential (Johnson & Kotz, 1970) is

(19) $$F(\xi) = \exp(-e^{-\xi})$$

and the density function

(20) $$f(\xi) = \exp(-\xi)\,\exp(-e^{-\xi}).$$

The mean is

(21) $$\mu(\xi) = \gamma$$

with γ = .5772156649… (Euler's constant, see Abramowitz & Stegun, 1965, p. 255), and the variance equals

(22) $$\sigma^2(\xi) = \frac{\pi^2}{6}.$$

If $\mathbf{X}_1$ and $\mathbf{X}_2$ are two independent and identically distributed random variables with distribution function *F*. the difference between these two is a random variable with the difference distribution

(23) $$D_F\left(\xi\right) = \int_{-\infty}^{+\infty} F(\xi + \zeta)\, dF(\zeta).$$

If the distribution of $\mathbf{X}_1$ and $\mathbf{X}_2$ is a double exponential, the differences are logistically distributed.

… and the resulting theory

The just mentioned results can now be applied to propose a slightly modified procedure of probabilistic production selection in ACT-R that guarantees equivalence with a choice probability system corresponding to Luce's choice axiom.

In any ACT-R model at any point in time, procedural memory consists of a set *R* of productions. An evaluation E_i is attached to each $i \in R$. Given a current goal *g*, the set $R(g) \subseteq R$ consists of all productions with a goal pattern matched by *g*. The no-selection option leading to a 'pop with failure' of the current goal is represented by o; its evaluation E_o is defined to be zero.

Let α be a real-valued and non-zero constant and $O(g) = \left\{\xi(i) \mid i \in \left(R(g) \cup \{o\}\right)\right\}$ a set of independent and double-exponentially distributed random variables with distribution function

(24) $$F(\xi) = \exp(-e^{-\xi/\alpha}).$$

If g is the current goal, select that option *i* for which

$$\left(E_i + \xi(i)\right) = \max\left\{\left(E_j + \xi(j)\right) \mid j \in O(g)\right\}.$$

The distribution function of the new random variables $\zeta(i) = E_i + \xi(i)$ equals

(25) $$F(\zeta) = \exp\left(-\exp\left(-\frac{\zeta - E}{\alpha}\right)\right) = \exp\left(-e^{-\zeta/\alpha} e^{E/\alpha}\right)$$

If and only if ζ has a maximum, $\tau = -e^{-\zeta/a}$ has a maximum too. This allows us to proceed further with the much simpler random variables τ that follow an exponential distribution and take densities larger zero for negative arguments only.

The probability of selecting option *i* equals the probability that $\tau(i)$ takes the maximum of all $\tau(j)$ and this is (according to equation (18)) equal to

(26) $$\begin{aligned} P(i,O(g)) &= \int_{-\infty}^{0} \left(e^{E_i/\alpha} e^{\frac{E_i}{\alpha}\tau} \left(\prod_{j \in O(g)-\{i\}} e^{\frac{E_j}{\alpha}\tau} \right)\right) d\tau \\ &= e^{E_i/\alpha} \int_{-\infty}^{0} \exp\left(\sum_{j \in O(g)} e^{E_j/\alpha} \tau \right) d\tau \\ &= \frac{e^{E_i/\alpha}}{\sum_{j \in O(g)} e^{E_j/\alpha}} \end{aligned}$$

To assimilate the formalism to the notion given in Anderson and Lebieres ‘Conflict Resolution Equation’, we propose to write *t* instead of α.

To make plain the main difference between this approach and Anderson and Lebiere’s, the special evaluation of the no-selection option should be made explicit. The resulting new form of Equation (26) is then

(27) $$P(i,g) = \frac{e^{E_i/t}}{1 + \sum_{j \in R(g)} e^{E_j/t}}$$

giving the probability of selecting production *i*, and

(28) $$P(o,g) = \frac{1}{1 + \sum_{j \in R(g)} e^{E_j/t}}$$

giving the probability of selecting no production and popping *g* with failure.

A reformulation of the random variable representation can be used to re-instantiate even the assumption of logistically distributed variables. As already mentioned, differences between double-exponentially distributed variables are themselves logistically distributed. Subtracting $\zeta(o)$ from each of the other $\zeta(i)$ results in a selection procedure, that adds a logistically distributed $\varepsilon(i)$ to each production evaluation and selects that production for which the resulting variable is the maximum over the set

$$\left\{0, E_1 + \varepsilon(1), E_2 + \varepsilon(2), \ldots\right\}.$$

The variance of each ε is $a^2 \frac{\pi^2}{3}$; the factor *a* introduced here is the same as the factor *s* in Anderson and Lebiere (1998, p. 64). The central difference is that in the approach proposed here, the $\varepsilon(i)$ are *not* independent. Since they all share the common component $\varepsilon(o)$, the covariance between two of them is always $a^2 \frac{\pi^2}{6}$ and the correlation is .50.

Under the selection assumption used here (Equations (11) and (12)), the ϕ-values have ration-scale quality; they are unique up to multiplication with a constant unequal zero. It follows immediately from Equation (27) that

(29) $$\phi(i) = e^{E_i/t}.$$

Therefore, E_i/t is at most unique up to the addition of a constant; it belongs to a difference scale or to a scale of higher type. E_o has been fixed at point zero. Therefore, the additive constant of an admissible transformation must be zero. This forces E_i/t to be even a value of an absolute scale. In ACT-R, E_i is a ‘*PG–C*’-value. *P* is a probability, its value has absolute-scale property, and the cost factor *C* is usually interpreted as an estimate of the time needed to reach the goal set by the evaluated production. To justify the subtraction of *C* from *PG*, *G* must be of dimension [time] too. It can be interpreted as the maximum time the system is willing to spend on the pursuit of the goal set by the production in question. Since E_i/t necessarily has to be an absolute-scale value, the factor t^{-1} must be of dimension [$time^{-1}$].

This has two consequences. At first, if the time scale used for C (and therefore implicitly for G too) is changed, the scale of t has to be altered accordingly. If for instance values of μsec are divided by 1,000 to switch to seconds, the *t*-value has to be divided by 1,000 too. The second consequence is that [$time^{-1}$] reminds of an expression for unit-speed that probably could be interpreted as something like the speed with which time needed runs in the system. If *t* is doubled, the time given in ‘*PG–C*’ is used up faster, its value is only half the original one. If we speculate that *t* represents something like ‘mental speed’ of time flow, it is worthwhile to look for inter- and intra-individual variation of t and to possible correlations with other speed-indicators more common in psychological research. But this is a problem requiring further research.

REFERENCES

Adams, E., & Messick, S. (1957). *An axiomatization of Thurstone's successive intervals and paired comparisons scaling models* (Tech. Report No. 12). Stanford University, Behavioral Sciences Division, Applied Mathematics and Statistics Laboratory, Stanford, CA.

Anderson, J. R., & Lebiere, C. (1998). *The atomic components of thought.* Lawrence Erlbaum, Mahwah NJ.

Bradley, R. A. (1954 a). Incomplete block rank analysis: On the appropriateness of the model for a method of pair comparisons. *Biometrics, 10,* 375–390.

Bradley, R. A. (1954 b). Rank analysis of incomplete block designs: II. Additional tables for the method of pair comparisons. *Biometrika, 41*, 502–537.

Bradley, R. A. (1955). Rank analysis of incomplete block designs: III. Some large-sample results on estimation and power for a method of paired comparisons. *Biometrika, 42*, 450–470.

Bradley, R. A., & Terry, M. E. (1952). Rank analysis of incomplete block designs: I. The method of pair comparisons. *Biometrika, 39*, 324–345.

Clarke, F. R. (1957). Constant-ratio rule for confusion matrices in speech communication. *Journal of the acoustical Society of America, 29*, 715–720.

Johnson, N. I., &Kotz, S. (1970). *Distributions in statistics: Continuous univariate distributions* (2 vols.). Houghton Mifflin, Boston MS.

Luce, R. D. (1959). *Individual choice behavior*. Wiley, New York NY.

Luce, R. D., & Galanter, E. (1963). Discrimination. In R. D. Luce, R. R. Bush & E. Galanter (eds.), *Handbook of mathematical psychology, Vol. I* (pp. 191–244). Wiley, New York NY.

Luce, R. D., & Suppes, P. (1965). Preference, utility, and subjective probability. In R. D. Luce, R. R. Bush & E. Galanter (eds.), *Handbook of mathematical psychology, Vol. III* (pp. 249–410). Wiley, New York NY.

Suppes, P., Krantz, D. M., Luce, R. D., & Tversky, A. (1989). *Foundations of measurement, Vol. II: geometrical, threshold, and probabilistic representation.* Academic Press, San Diego CA.

Suppes, P., & Zinnes, J. L. (1963). Basic measurement theory. In R. D. Luce, R. R. Bush & E. Galanter (eds.), *Handbook of mathematical psychology, Vol. I* (pp. 1–76). Wiley, New York NY.

Thurstone, L. L. (1927). *A law of comparative judgement.* Psychological Review, 34, 273–286.

Yellott, J. I. (1977). The relationship between Luce‘s choice axiom, Thurstone‘s theory of comparative judgement, and the double exponential distribution. *Journal of Mathematical Psychology, 15*, 109–144.